The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3

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Classification AMS. 35 J 65

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Abstract

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1 Introduction and results

Let us consider the nonlinear elliptic problem

$$(P)\begin{cases} -\Delta u &= u^p \ , \ u > 0 \quad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N, N \geq 3$, and $p = \frac{N+2}{N+2}$ is the critical Sobolev exponent. It has been known, since a work by Pohozaev [15] that (P) has no solution when Ω is starshaped. On the other hand, Kazdan and Warner [10] proved that a solution exists in the special case where Ω is an annulus. A major step was accomplished by Bahri and Coron [2] who showed that (P) has a solution as soon as Ω has nontrivial topology, in the sense that $H_{2n-1}(\Omega;\mathbb{Q}) \neq 0$ or $H_n(\Omega;\mathbb{Z}/2\mathbb{Z}) \neq 0$ for some $n \in \mathbb{N}^*$. However, contractible domains have been exhibited for which (P) admits also a solution [8][9], pointing out that the geometry of the domain is of some importance.

A strategy to prove existence or nonexistence of solutions to (P) is to compute the difference of topology between the level sets of a functional associated to the problem. However, such a functional is noncompact : critical points at infinity occur, that is orbits of the gradient along which the functional remains bounded, its gradient goes to zero, and which do not converge. In view of further results of existence, nonexistence or multiplicity concerning (P), it is therefore crucial to know the exact contribution of the critical points at infinity to the relative topology between the level sets. This program was performed in [3], adopting the following strategy. (P) is approximated by the subcritical problems

$$(P_\varepsilon)\left\{\begin{array}{lll} -\Delta u & = u^{p-\varepsilon}\ ,\ u>0 & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega \end{array}\right.$$

where ε is a strictly positive small real number. To (P_ε) is associated the functional

$$(1) \qquad J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1-\varepsilon}\int_\Omega |u|^{p+1-\varepsilon} \qquad u \in H_0^1(\Omega)$$

which is compact, and whose positive critical points are solutions to (P_ε). Such solutions exist and it follows from [12] [20] that, as ε goes to zero, either they converge to a solution to (P), or they blow up at a finite number of points in Ω. Namely, (u_ε) being a bounded sequence in $H_0^1(\Omega)$ of solutions to (P_ε), up to a subsequence we have :

$$(2) \qquad u_\varepsilon = u_0 + \sum_{i=1}^{k} \alpha_i^\varepsilon P\delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + v^\varepsilon$$

with u_0 a solution to (P) or $u_0 \equiv 0$, v^ε goes to zero in $H_0^1(\Omega)$, $k \in \mathbb{N}$. Moreover, for $\lambda \in \mathbb{R}_+^*$, $x \in \mathbb{R}^N$

$$(3) \qquad \begin{array}{llll} \delta_{\lambda,x}: & \mathbb{R}^N & \to \mathbb{R} \\ & y & \mapsto \lambda^{\frac{N-2}{2}}(1+\lambda^2|x-y|^2)^{-\frac{N-2}{2}} \end{array}$$

and $P\delta_{\lambda,x}$ is defined as the projection of $\delta_{\lambda,x}$ onto $H_0^1(\Omega)$, that is :

$$(4) \qquad \Delta P\delta_{\lambda,x} = \Delta\delta_{\lambda x} \text{ in } \Omega\ ;\ P\delta_{\lambda,x} = 0 \text{ on } \partial\Omega\ .$$

Lastly

$$\begin{array}{ll} \alpha_i^\varepsilon \in \mathbb{R} \, , \alpha_i^\varepsilon & \to \bar{\alpha} = (N(N-2))^{\frac{N-2}{4}} \\ x_i^\varepsilon \in \Omega \, , x_i^\varepsilon & \to x_i \in \overline{\Omega} \end{array}$$

and

$$(5) \qquad \left\{ \begin{array}{l} \lambda_i^\varepsilon d(x_i^\varepsilon, \partial\Omega) \to +\infty \\ \\ \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon \lambda_j^\varepsilon |x_i^\varepsilon - x_j^\varepsilon|^2 \to +\infty \, . \end{array} \right.$$

We note that the functions $\bar{\alpha}\delta_{\lambda x}, \lambda > 0 \, , x \in \mathbb{R}^N$, are the only solutions of the problem

$$-\Delta u = u^p \ , \ u > 0 \text{ in } \mathbb{R}^N$$

see [6] - and that $P\delta_{\lambda,x} \ , \ \lambda > 0 \, , x \in \Omega$, satisfies

$$(6) \qquad \left\{ \begin{array}{ll} -\Delta P\delta_{\lambda,x} & = \delta_{\lambda,x}^p \text{ in } \Omega \\ P\delta_{\lambda,x} & = 0 \text{ on } \partial\Omega \end{array} \right.$$

(5) ensures that we have

$$(7) \qquad J_\varepsilon(u_\varepsilon) = J_0(u_0) + k\frac{S^{N/2}}{N} + o(1)$$

where $S = \inf_{u \in H_0^1(\Omega), u \neq 0} |u|^2_{H_0^1(\Omega)} |u|^{-2}_{L^{p+1}(\Omega)}$ is the Sobolev constant, which depends on N only. Actually, a result by Schoen [19] says that, under the previous assumptions, we have either $k = 0$ or $u_0 \equiv 0$. If $k = 0$, u_ε converges in $H_0^1(\Omega)$ to a solution of (P) ; if $u_0 \equiv 0$, u_ε blows up as ε goes to zero, namely

$$(8) \qquad \left\{ \begin{array}{ll} |\nabla u_\varepsilon|^2, u_\varepsilon^{p+1} \underset{\varepsilon \to 0}{\rightharpoonup} S^{N/2} \sum_{i=1}^k \delta_{x_i} & \qquad (\delta_{x_i} \text{ is the Dirac mass at } x_i) \\ J_\varepsilon(u_\varepsilon) \to c_k = k\frac{S^{N/2}}{N}. & \end{array} \right.$$

Let us denote by G and H respectively the Green's function of the Laplacian with Dirichlet boundary conditions on Ω, and its regular part, i.e.

$$(9) \qquad \left\{ \begin{array}{ll} G(x,y) = \frac{1}{|x-y|^{N-2}} - H(x,y) & \qquad (x,y) \in \Omega \times \Omega \\ \Delta_x H = 0 \text{ in } \Omega \times \Omega & \qquad G = 0 \text{ on } \partial(\Omega \times \Omega) \, . \end{array} \right.$$

For $k \in \mathbb{N}^*$ and $\mathbf{x} = (x_1, \cdots, x_k) \in \Omega^k$, we set $M(\mathbf{x}) = (m_{ij})_{1 \le i,j \le k}$ the matrix defined as

$$(10) \qquad m_{ii} = H(x_i, x_i) \, ; m_{ij} = -G(x_i, x_j) \qquad i \neq j$$

and $\rho(\mathbf{x})$ denotes the least eigenvalue of M $\quad(\rho(\mathbf{x})=-\infty$ if $x_i=x_j$ for some $i\neq j)$. We define also

$$(11) \qquad \begin{array}{lll} F_{\mathbf{x}}: & (\mathbb{R}_+^*)^k & \to \mathbb{R} \\ & \Lambda=(\Lambda_1,\cdots,\Lambda_k) & \mapsto \frac{1}{2}\Lambda M(\mathbf{x})^t\Lambda-\sum_{i=1}^k \ell n\Lambda_i \,. \end{array}$$

If $\rho(\mathbf{x})>0$, $F_{\mathbf{x}}$ being strictly convex in $(\mathbb{R}_+^*)^k$ and infinite on the boundary, $F_{\mathbf{x}}$ has a unique critical point $\Lambda(\mathbf{x})$, which is a minimum. On

$$(12) \qquad \rho^+=\{\mathbf{x}\in\Omega^k/\rho(\mathbf{x})>0\}$$

we set

$$(13) \qquad \widetilde{F}(\mathbf{x})=F_{\mathbf{x}}(\Lambda(\mathbf{x}))=\frac{k}{2}-\sum_{i=1}^k \ell n\Lambda_i(\mathbf{x})$$

whose differential is given by

$$(14) \qquad \widetilde{F}'(\mathbf{x})=\frac{1}{2}\Lambda(\mathbf{x})M'(\mathbf{x})^t\Lambda(\mathbf{x})=-\sum_{i=1}^k\frac{\Lambda_i'(\mathbf{x})}{\Lambda_i(\mathbf{x})}\,.$$

In [3] it is proved :

Theorem 1 *Assume that $N\geq 4$, and (u_ε) is a sequence of solutions to (P_ε) which blows up at k points $x_1,\cdots,x_k$ of $\overline{\Omega}$ as ε goes to zero. Then*

(a) $\mathbf{x}=(x_1,\cdots,x_k)\in\Omega_{d_0}^k$, *with* $d_0=d_0(\Omega)>0$ *and* $\Omega_{d_0}=\{x\in\Omega/d(x,\partial\Omega)>d_0\}$

(b) $\rho(\mathbf{x})\geq 0$

(c) either $\rho(\mathbf{x})>0$ *and* $\widetilde{F}'(\mathbf{x})=0$, *or* $\rho(\mathbf{x})=0$ *and* $\rho'(\mathbf{x})=0$.

Conversely, if $\mathbf{x}\in\rho^+$ is a nondegenerate critical point of $\widetilde{F}$, there exists for ε small enough a sequence of solutions to (P_ε) which blows up at $x_1,\cdots,x_k$ as ε goes to zero.

Assuming that zero is a regular value for ρ, (b) and (c) may be replaced by

(b') $\rho(x)\geq\rho_0(\Omega)>0$

(c') $\widetilde{F}'(\mathbf{x})=0$.

Moreover, if $\rho(\mathbf{x}) > 0$, the following estimates hold :

$$\frac{1}{(\lambda_i^\varepsilon)^{\frac{N-2}{2}}} \sim \gamma \Lambda_i(\mathbf{x}) \varepsilon^{1/2} \; ; \; J_\varepsilon(u_\varepsilon) = c_k + k\gamma_1 \varepsilon |\ell n \varepsilon| + k\gamma_2 \varepsilon + 2\gamma_1 \varepsilon \widetilde{F}(\mathbf{x}) + o(\varepsilon)$$

$\gamma > 0 \, , \gamma_1 > 0 \, , \gamma_2$ are constants which depend on N only.

Using these results it is showed, denoting by

$$J_\varepsilon^c = \{u \in H_0^1(\Omega) / J_\varepsilon(u) < c\}$$

the level sets of J_ε :

Theorem 2 *Assume that $N \geq 4$, and zero is a regular value of ρ. The contribution to the relative homology*

$$H_* \left(J_\varepsilon^{c_k+\eta}, J_\varepsilon^{c_k-\eta} \right)$$

of the solutions to (P_ε) which blow up k times as ε goes to zero, $0 < \eta < \frac{S^{N/2}}{N}$, is equal for ε small enough to

$$H_* \left((\Omega^k, \rho^-) \underset{\sigma_k}{\times} (D^k, S^{k-1}) \right)$$

with

$$\rho^- = \{\mathbf{x} \in \Omega^k / \rho(\mathbf{x}) \leq 0\} \; .$$

As the result does not depend on ε for ε sufficiently small, intercalating the level sets of J_0 between the level sets of J_ε, it provides us with the contribution of the critical points at infinity to the relative homology between the level sets of J_0.

It is to be noticed that the relative topology (Ω^k, ρ^-) has been computed in the particular case of thin or expanding annuli-domains [13] [14]. For large enough annuli, this difference turns out to be zero for $k \geq 2$, whereas for thin enough annuli, this difference is nonzero, at least for $k = 2$. This indicates that some solutions of Yamabe-type equations may transform into critical points at infinity or, conversely, be produced by bifurcation from infinity (an example of such a phenomenon is given in [16]), and this possible link shows that a careful study of the critical points at infinity is necessary to get a good understanding in this kind of problems.

The aim of the paper is to show that the results of [3] are true in dimension 3, namely :

Theorem 3 *Conclusions of Theorem 1 and 2 hold for $N = 3$.*

In fact, the only thing we have to prove is that the statements of Theorem 1 hold in dimension 3, since the statement of Theorem 2 follows exactly in the same way as in [3, Section 5] (for $N \geq 4$, one can also use the Morse lemma at infinity established in [4]). Concerning Theorem 1, the strategy is the same as in higher dimensions, but we need a more careful analysis of the properties of the v-part of u which occurs in (2).

The next section is devoted to the general setting, which will allow us to prove, in Section 3, that the conclusions of Theorem 1 hold in dimension 3. Section 4 provides us with the estimates required to justify the previous arguments.

Remark The extension to dimension 3 of results previously known in higher dimensions relies on improved estimates of some integral quantities. Using the same techniques we can also extend to dimension 4 the conclusion of Theorem 2 in [17], previously known for $N \geq 5$, namely :

Theorem 4 *Assume that $N \geq 4$, and x_0 is a nondegenerate critical point of $\varphi(x) = H(x, x)$. For $\varepsilon > 0$ small enough, problem*

$$(P'_\varepsilon)\begin{cases} -\Delta u &= u^p + \varepsilon u \ , \ u > 0 \quad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

has a solution u_ε which blows up at x_0 as ε goes to zero, that is $|\nabla u_\varepsilon|^2 \underset{\varepsilon \to 0}{\rightharpoonup} S^{N/2}\delta_{x_0}$.

In this case, the assumption $N \geq 4$ is sharp, since in dimension 3 (P'_ε) has no solution for small ε when Ω is starshaped [5].

2 The setting

From now on, we assume that $N = 3$. For $a > 0$, we define the subset of $H_0^1(\Omega)$

$$F_a = \left\{ \alpha \sum_{i=1}^{k} P\delta_{\lambda_i, x_i} / \lambda d(x_i, \partial\Omega) > \frac{1}{a}, \forall i \, ; \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j|^2 > \frac{1}{a}, \forall i, j, i \neq j \right\}$$

where $\bar{\alpha}$ $(= 3^{1/4})$ and $P\delta_{\lambda,x}$ are defined in Section 1. According to (3) (4), we may write

$$P\delta_{\lambda,x} = \delta_{\lambda,x} - \varphi_{\lambda,x} \tag{15}$$

with

$$(16) \qquad \Delta\varphi_{\lambda,x} = 0 \text{ in } \Omega\,;\, \varphi_{\lambda,x} = \delta_{\lambda,x} \text{ on } \partial\Omega$$

and from the maximum principle we deduce :

$$(17) \qquad \varphi_{\lambda,x}(y) = \frac{1}{\lambda^{1/2}} H(x,y) + 0\left(\frac{1}{\lambda^{3/2} d(x,\partial\Omega)}\right) .$$

(u_ε) being a sequence of solutions to (P_ε) which blows up at k points (not necessarily distinct) of $\overline{\Omega}$ as ε goes to zero, it follows from Section 1 that the distance in $H_0^1(\Omega)$-norm between u_ε and F_a goes to zero with ε. Since we know from [2, Proposition 7] that, provided that $dist_{H_0^1(\Omega)}(u\ , F_a) < a$, with a sufficiently small, the problem

$$Minimize\ |u - \sum_{i=1}^{k} \alpha_i P\delta_{\lambda_i,x_i}|_{H_0^1}\ with\ respect\ to\ the\ \alpha_i, \lambda_i, {x_i}'s$$

has a unique solution in the open set

$$|\alpha_i - \bar{\alpha}| < 4a, \lambda_i d(x_i,\partial\Omega) > \frac{1}{4a}\ , \forall i\,;\, \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|x_i - x_j|^2 > \frac{1}{4a}, \forall i,j, i \neq j\ ,$$

we can easily prove the existence of a diffeomorphism between a neighbourhood of the eventual solutions to (P_ε) which concentrate k times as ε goes to zero and the open set

$$M = \Big\{ m = (\alpha, \lambda, \mathbf{x}, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times \Omega^k \times H_0^1(\Omega) /$$
$$|\alpha_i - \bar{\alpha}| < \nu_0\ ,\ \lambda_i d(x_i,\partial\Omega) > \frac{1}{\nu_0},\ \forall i\ ;$$
$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_i}{\lambda_i} + \lambda_i\lambda_j|x_i - x_j|^2 > \frac{1}{\nu_0}\ , \forall i,j,\ i \neq j ; v \in E_{\lambda,\mathbf{x}}, |v|_{H_0^1} < \nu_0 \Big\}$$

with $\nu_0 > 0$ some suitable constant, and (here and throughout the sequel, $P\delta i$ denotes $P\delta_{\lambda_i,x_i}$)

$$(18) \qquad E_{\lambda,\mathbf{x}} = \Big\{ v \in H_0^1(\Omega) / \langle v, P\delta_i \rangle_{H_0^1} = \langle v, \tfrac{\partial P\delta_i}{\partial \lambda_i} \rangle_{H_0^1} = \langle v, \tfrac{\partial P\delta_i}{\partial (x_i)_j} \rangle_{H_0^1} = 0\ ,\ 1 \leq i \leq k\ ,\ 1 \leq j \leq 3 \Big\} .$$

We know that u_ε, solution to (P_ε) blowing up at k points as ε goes to zero, may be written as

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_i^\varepsilon P\delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + v^\varepsilon \tag{19}$$

for ε small enough, with $m^\varepsilon = (\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon) \in M$ and $\alpha_i^\varepsilon \to \bar{\alpha}$,

$$\lambda_i^\varepsilon d(x_i^\varepsilon, \partial\Omega) \to +\infty \;, \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon \lambda_j^\varepsilon |x_i^\varepsilon - x_j^\varepsilon|^2 \to +\infty \;, v^\varepsilon \to 0 \;.$$

Moreover, it follows from an argument by Z.C. Han [9] that the concentration points cannot approach the boundary, that is there exists $d_0 = d_0(\Omega) > 0$ such that for ε small enough

$$d(x_i^\varepsilon, \partial\Omega) > d_0 \quad 1 \le i \le k \;.$$

Schoen [19] also showed that there exist $d_0'(\Omega) > 0$ and $c_0(\Omega) > 0$ such that for ε small enough

$$|x_i^\varepsilon - x_j^\varepsilon| > d_0' \;, \; \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} < c_0 \quad 1 \le i \,, j \le k \;, \; i \ne j \;.$$

A proof of the first property follows from Y.Y. Li's work about the scalar curvature problem [11], showing that isolated blow up points are in fact isolated simple blow up points. The second property, concerning the concentration parameters, makes the computations easier, but could be recovered by our own analysis later.

Lastly, multiplying the equation $-\Delta u_\varepsilon = u_\varepsilon^{p-\varepsilon}$ by $P\delta_{\lambda_i^\varepsilon, x_i^\varepsilon}$ and integrating on Ω, we obtain, using the expansion (19), together with the listed properties of $\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon, v^\varepsilon$ and integral estimates in [18]

$$C + o(1) = \frac{C}{(\lambda_i^\varepsilon)^{1/2}} + o(1)$$

with $C = \bar{\alpha} \int_{\mathbb{R}^3} |\nabla \delta_{\lambda,x}|^2 = 3^{-1/4} S^{3/2}$. Therefore

$$\varepsilon \ell n \lambda_i^\varepsilon \to 0 \text{ as } \varepsilon \to 0 \quad 1 \le i \le k \;.$$

Collecting these informations, we conclude that there exists a diffeomorphism between a neighbourhood of the eventual solutions to (P_ε) with k peaks and the open set

$$M_\varepsilon = \Big\{ m = (\alpha, \lambda, \mathbf{x}, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times \Omega_{d_0}^k \times H_0^1(\Omega) /$$

$$|\alpha_i - \alpha| < \nu_0 \ , \ \lambda_i > \frac{1}{\nu_0}, \ \varepsilon \ell n \lambda_i < \nu_0 \ , 1 \le i \le k \ ;$$

$$\frac{\lambda_i}{\lambda_j} < c_0 \ , \ |x_i - x_j| > d_0' \ , 1 \le i, j \le k \ , i \ne j; v \in E_{\lambda, \mathbf{x}}, |v|_{H_0^1} < \nu_0 \Big\}$$

ν_0, c_0, d_0, d_0' being some strictly positive constante suitably chosen. Defining on M_ε the functional

$$(20) \qquad \begin{array}{rll} K_\varepsilon : & M_\varepsilon & \to \mathbb{R} \\ & m = (\alpha, \lambda, \mathbf{x}, v) & \mapsto J_\varepsilon(\sum_{i=1}^k \alpha_i P\delta_{\lambda_i, x_i} + v) \end{array}$$

we have

Proposition 1 *$m = (\alpha, \lambda, \mathbf{x}, v) \in M_\varepsilon$ is a critical point of K_ε if and only if $u = \sum\limits_{i=1}^k \alpha_i P\delta_{\lambda_i, x_i} + v$ is a critical point of J_ε. This means that there exists $(A, B, C,) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ such that*

$$(E) \left\{ \begin{array}{lll} (E_{\alpha_i}) & \frac{\partial K_\varepsilon}{\partial \alpha_i} = 0 \qquad 1 \le i \le k & \\ (E_{\lambda_i}) & \frac{\partial K_\varepsilon}{\partial \lambda_i} = B_i \langle \frac{\partial^2 P\delta_i}{\partial \lambda_i^2}, v \rangle_{H_0^1} + \sum\limits_{j=1}^3 C_{ij} \langle \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial (x_i)_j}, v \rangle_{H_0^1} & 1 \le i \le k \\ (E_{(x_i)_j}) & \frac{\partial K_\varepsilon}{\partial (x_i)_j} = B_i \langle \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial (x_i)_j}, v \rangle_{H_0^1} + \sum\limits_{\ell=1}^3 C_{i\ell} \langle \frac{\partial^2 P\delta_i}{\partial (x_i)_j \partial (x_i)_\ell}, v \rangle_{H_0^1} & {\scriptstyle 1 \le i \le k \atop 1 \le j \le 3} \\ (E_v) & \frac{\partial K_\varepsilon}{\partial v} = \sum\limits_{i=1}^k \Big(A_i P\delta_i + B_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum\limits_{j=1}^3 C_{ij} \frac{\partial P\delta_i}{\partial (x_i)_j} \Big) \ . & \end{array} \right.$$

3 Proof of Theorem 1 for $N = 3$

3.1 The v-part of u

We first look at the last equation of (E), concerning the derivative of K_ε with respect to v. The result is the following :

Proposition 2 *There exists a smooth map which to any $(\varepsilon, \alpha, \lambda, \mathbf{x})$ such that ε is small enough and $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$, associates $\bar{v} \in E_{\lambda, \mathbf{x}}, |\bar{v}|_{H_0^1} < \nu_0$, such that (E_v) is satisfied for some $(A, B, C) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$. Such a $\bar{v}$ is* *unique, minimizes* $K_\varepsilon(\alpha,\lambda,\mathbf{x},v)$ *with respect to* v *in* $\{v \in E_{\lambda,\mathbf{x}}/|v|_{H_0^1} < \nu_0\}$, *and we have the estimates*

$$|\bar{v}|_{H_0^1} = 0(\varepsilon + \frac{1}{\lambda}) \tag{21}$$

$$\left\{ \begin{array}{ll} A & = 0(|\beta| + \frac{1}{\lambda} + \varepsilon \ell n \lambda + \varepsilon^2) \\ B & = 0(1+\varepsilon\lambda) \\ C & = 0(\frac{1}{\lambda^3} + \frac{\varepsilon^2}{\lambda}) \end{array} \right. \tag{22}$$

with

$$\beta = (\beta_1, \cdots, \beta_k) = (\bar{\alpha} - \alpha_1, \cdots, \bar{\alpha} - \alpha_k) \ . \tag{23}$$

For sake of simplicity, $\frac{\lambda_i}{\lambda_j}$ being bounded for any i, j, $0(f(\lambda))$ denotes any quantity dominated by $\sum\limits_{i=1}^{k} f(\lambda_i)$.

We sketch the proof of Proposition 2. Expanding K_ε with respect to v, we obtain

$$K_\varepsilon(\alpha,\lambda,\mathbf{x},v) = K_\varepsilon(\alpha,\lambda,\mathbf{x},0) - \int_\Omega (\sum_{i=1}^{k} \alpha_i P\delta_i)^{5-\varepsilon} v + \frac{1}{2}\int_\Omega |\nabla v|^2$$

$$- \frac{5-\varepsilon}{2} \int_\Omega (\sum_{i=1}^{k} \alpha_i P\delta_i)^4 v^2 + R_{\varepsilon,\alpha,\lambda,\mathbf{x}}(v)$$

with

$$R_{\varepsilon,\alpha,\lambda,\mathbf{x}}(v) = 0(|v|^3_{H_0^1}), R'_{\varepsilon,\alpha,\lambda,\mathbf{x}}(v) = 0(|v|^2_{H_0^1}), R''_{\varepsilon,\alpha,\lambda,\mathbf{x}}(v) = 0(|v|_{H_0^1})$$

uniformly with respect to $\varepsilon, \alpha, \lambda, \mathbf{x}$, $(\alpha,\lambda,\mathbf{x},0) \in M_\varepsilon$ and ε small enough. Moreover, we know that the quadratic term in v is coercive, with a modulus of coercivity bounded form below as $(\alpha,\lambda,\mathbf{x},0) \in M_\varepsilon$ and ε is sufficiently small - for a proof of this fact, see [1][17][18]. We claim that

$$\int_\Omega (\sum_{i=1}^{k} \alpha_i P\delta_i)^{5-\varepsilon} v = 0\left((\varepsilon + \frac{1}{\lambda})|v|_{H_0^1} \right) \ . \tag{24}$$

Consequently, the implicit function theorem yields the conclusion of Proposition 2, together with estimate (21). Let us prove the claim. Since $v \in E_{\lambda,\mathbf{x}}$

and $P\delta_i$ satisfies (6), we may write

$$\begin{aligned}\int_\Omega (\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} v &= \int_\Omega \left((\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right) v \\ &= 0\left(\left| (\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|_{\frac{6}{5}} |v|_{H_0^1} \right)\end{aligned}$$

using Hölder's inequality and Sobolev embedding theorem. Clearly, far from the concentration points x_i, say outside $\bigcup_{i=1}^k B(x_i, d)$, with $d = \min(d_0, \frac{d_0'}{2})$,

$$\delta_i, P\delta_i = 0(\frac{1}{\lambda^{1/2}}), \text{ and } \left| (\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right| = 0(\frac{1}{\lambda^{5/2}}) .$$

On $B_j = B(x_j, d)$, we write

$$(\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 = \alpha_j^{5-\varepsilon} \left(\delta_j^{5-\varepsilon} - \frac{\delta_j^5}{\lambda_j^{\varepsilon/2}} \right) + 0(\frac{\delta_j^4}{\lambda^{1/2}})$$

since in this subdomain $P\delta_i, \varphi_j (= \varphi_{\lambda_j, x_j}) = 0(\frac{1}{\lambda^{1/2}}), i \neq j$, and $\delta_j \geq \frac{C}{\lambda^{1/2}}$ for some strictly positive constant C. (We note that, $\varepsilon \ell n \lambda_i$ being small, a quantity as δ_i^ε is close to 1 uniformly in Ω, for any i.) We remark that

$$\begin{aligned}&\left(\delta_j^{-\varepsilon} - \frac{1}{\lambda_j^{\varepsilon/2}} \right)(x) \\ &= \exp(-\varepsilon \ell n \delta_j(x)) - \exp(-\frac{\varepsilon}{2} \ell n \lambda_j) \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n \varepsilon^n}{2^n n!} \left((\ell n \lambda_j - \ell n(1 + \lambda_j^2 |x - x_j|^2))^n - (\ell n \lambda_j)^n \right) \\ &= 0\left(\sum_{n=1}^{+\infty} \frac{\varepsilon^n}{n!} \left((\ell n \lambda_j)^{n-1} \ell n(1 + \lambda_j^2 |x - x_j|^2) + (\ell n(1 + \lambda_j^2 |x - x_j|^2))^n \right) \right) \\ &= 0\left(\varepsilon \ell n(1 + \lambda_i^2 |x - x_j|^2) \right).\end{aligned}$$

Therefore, on B_j

$$\left| (\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|^{\frac{6}{5}} = 0\left(\varepsilon^{6/5} (\ell n(1 + \lambda_j^2 |x - x_j|^2))^{6/5} \delta_j^6 + \frac{\delta_j^{24/5}}{\lambda^{3/5}} \right)$$

As we have, performing the change of variable $y = (x - x_j)$ and using spherical coordinates

$$\int_{B_j} (\ell n(1 + \lambda_j^2 |x - x_j|^2))^{6/5} \delta_j^6 dx = 4\pi \int_0^{\lambda d} (\ell n(1 + r^2))^{6/5} \frac{r^2 dr}{(1 + r^2)^3} = 0(1)$$

and

$$\int_{B_j} \delta_j^{24/5} dx = \frac{4\pi}{\lambda^{3/5}} \int_0^{\lambda d} \frac{r^2 dr}{(1 + r^2)^{\frac{12}{5}}} = 0(\frac{1}{\lambda^{3/5}})$$

we finally obtain

$$\int_\Omega \left| (\sum_{i=1}^k \alpha_i P\delta_i)^{5-\varepsilon} - \sum_{i=1}^k \frac{\alpha_i^{5-\varepsilon}}{\lambda_i^{\varepsilon/2}} \delta_i^5 \right|^{\frac{6}{5}} = 0(\varepsilon^{6/5} + \frac{1}{\lambda^{6/5}})$$

hence the claim.

To complete the proof of Proposition 2, it only remains to show that estimate (22) holds. We proceed as follows : we take the scalar product in $H_0^1(\Omega)$ of (E_v) with $P\delta_i, \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{\partial P\delta_i}{\partial (x_i)_j}$ respectively, $1 \leq i \leq k, 1 \leq j \leq 3$. On the righ hand side, we get a linear system involving the quantities A_i, B_i, C_{ij}, which is nearly diagonal, invertible, and whose coefficients are given by

$$(25) \qquad \begin{cases} \int_\Omega \nabla P\delta_i \cdot \nabla P\delta_j = \Gamma_1 \delta_{ij} + 0(\frac{1}{\lambda}) \\ \int_\Omega \nabla \frac{\partial P\delta_i}{\partial \lambda_i} \cdot \nabla \frac{\partial P\delta_j}{\partial \lambda_j} = \frac{\Gamma_2}{\lambda_i^2} \delta_{ij} + 0(\frac{1}{\lambda^3}) \\ \int_\Omega \nabla \frac{\partial P\delta_i}{\partial (x_i)_a} \nabla \frac{\partial P\delta_j}{\partial (x_j)_b} = \Gamma_3 \lambda_i^2 \delta_{ij} \delta_{ab} + 0(\frac{1}{\lambda}) \end{cases}$$

δ_{ij}, δ_{ab} denoting the Kronecker symbol and $\Gamma_1, \Gamma_2, \Gamma_3$ being strictly positive constants, and

$$(26) \qquad \begin{cases} \int_\Omega \nabla P\delta_i \cdot \nabla \frac{\partial P\delta_j}{\partial \lambda_j} = 0(\frac{1}{\lambda^2}) \ ; \\ \int_\Omega \nabla P\delta_i \cdot \nabla \frac{\partial P\delta_j}{\partial (x_j)_a} = 0(\frac{1}{\lambda}) \ ; \\ \int_\Omega \nabla \frac{\partial P\delta_i}{\partial \lambda_i} \cdot \nabla \frac{\partial P\delta_j}{\partial (x_j)_a} = 0(\frac{1}{\lambda^2}) \ . \end{cases}$$

These estimates follow easily from (3)(4)(17), and may also be found in [1][17]. On the left hand side we find

$$\langle \frac{\partial K_\varepsilon}{\partial v}, P\delta_i \rangle_{H_0^1} = \frac{\partial K_\varepsilon}{\partial \alpha_i}; \langle \frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial \lambda_i} \rangle_{H_0^1} = \frac{1}{\alpha_i} \frac{\partial K_\varepsilon}{\partial \lambda_i}; \langle \frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial (x_i)_j} \rangle_{H_0^1} = \frac{1}{\alpha_i} \frac{\partial K_\varepsilon}{\partial (x_i)_j}$$

and we have :

Proposition 3 *For ε small enough and $(\alpha, \lambda, \mathbf{x}, 0) \in M_\varepsilon$, the following estimates hold*

$$\frac{\partial K_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, \mathbf{x}, \bar{v}) = -K_1\beta_i + V_{\alpha_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) \tag{27}$$

with V_{α_i} a smooth function which satisfies

$$V_{\alpha_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0(\beta_i^2 + \frac{1}{\lambda} + \varepsilon \ell n\lambda + \varepsilon^2);$$

$$\begin{aligned} \tfrac{\partial K_\varepsilon}{\partial \lambda_i}(\alpha, \lambda, \mathbf{x}, \bar{v}) = {} & \tfrac{1}{\lambda_i}\left(K_2\varepsilon - K_3\left(\tfrac{H(x_i,x_i)}{\lambda_i} - \sum_{\ell\neq i}\tfrac{G(x_i,x_\ell)}{\lambda_i^{1/2}\lambda_\ell^{1/2}}\right)\right) \\ & + V_{\lambda_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) \end{aligned} \tag{28}$$

with V_{λ_i} a smooth function which satisfies

$$V_{\lambda_i}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0\left(\frac{\ell n\lambda}{\lambda^4} + \varepsilon^2\frac{\ell n\lambda}{\lambda} + |\beta|(\frac{\varepsilon}{\lambda} + \frac{1}{\lambda^2})\right);$$

$$\begin{aligned} \tfrac{\partial K_\varepsilon}{\partial (x_i)_j}(\alpha, \lambda, \mathbf{x}, \bar{v}) = {} & K_4\left(\tfrac{1}{\lambda_i}\tfrac{\partial H}{\partial a_j}(x_i, x_i) - \sum_{\ell\neq i}\tfrac{1}{\lambda_i^{1/2}\lambda_j^{1/2}}\tfrac{\partial G}{\partial a_j}(x_i, x_\ell)\right) \\ & + V_{(x_i)_j}(\varepsilon, \alpha, \lambda, \mathbf{x}) \end{aligned} \tag{29}$$

where $\frac{\partial}{\partial a_j}$ (resp. $\frac{\partial}{\partial b_j}$) *denotes the derivative with respect to the j-th component of the first (resp. second) variable, and $V_{(x_i)_j}$ is a smooth function which satisfies*

$$V_{(x_i)_j}(\varepsilon, \alpha, \lambda, \mathbf{x}) = 0\left(\frac{1}{\lambda^{3/2}} + \lambda^{1/2}\varepsilon^2 + \frac{|\beta|}{\lambda}\right);$$

K_1, K_2, K_3, K_4 are strictly positive constants.

The proof of this proposition is delayed until the next section.

Proof of Proposition 2 completed. From Proposition 3 we deduce that

$$\left\{\begin{array}{ll} \frac{\partial K_\varepsilon}{\partial \alpha_i} & = 0(|\beta| + \frac{1}{\lambda} + \varepsilon\ell n\lambda + \varepsilon^2) \\ \frac{\partial K_\varepsilon}{\partial \lambda_i} & = 0\left(\frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda}\right) \\ \frac{\partial K_\varepsilon}{\partial (x_i)_j} & = 0\left(\frac{1}{\lambda} + \lambda\varepsilon^2\right) \end{array}\right. \tag{30}$$

hence (22), using (25)(26) to invert the linear system involving A_i, B_i, C_{ij}.

Once (E_v) is solved, we are left with a finite dimensional system of equations $(E_{\alpha_i}), (E_{\lambda_i}), (E_{(x_i)_j}), 1 \leq i \leq k, 1 \leq j \leq 3$, whose left hand side is given by Proposition 3, and whose righ hand side may be estimated using (22), namely

$$\begin{aligned} B_i\langle \tfrac{\partial^2 P\delta_i}{\partial\lambda_i^2}, \bar v\rangle_{H_0^1} + \textstyle\sum_{j=1}^3 C_{ij}\langle \tfrac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}, \bar v\rangle_{H_0^1} &= 0\left(\left(\tfrac{|B_i|}{\lambda} + \textstyle\sum_{j=1}^3 |C_{ij}|\right)|\bar v|_{H_0^1}\right) \\ &= 0\left(\left(\tfrac{1}{\lambda^2} + \tfrac{\varepsilon}{\lambda}\right)|\bar v|_{H_0^1}\right) \end{aligned}$$

since, as straight forward computations show

$$(31) \qquad \left|\frac{\partial^2 P\delta_i}{\partial\lambda_i^2}\right|_{H_0^1} = 0(\frac{1}{\lambda^2}) \qquad \left|\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}\right|_{H_0^1} = 0(1);$$

in the same way

$$\begin{aligned} B_i\langle \tfrac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}, \bar v\rangle_{H_0^1} + \textstyle\sum_{\ell=1}^3 C_{i\ell}\langle \tfrac{\partial^2 P\delta_i}{\partial(x_i)_j\partial(x_i)_\ell}, \bar v\rangle_{H_0^1} &= 0\left(\left(\tfrac{|B_i|}{\lambda^{1/2}} + \sum_{\ell=1}^3 \lambda^2|C_{i\ell}|\right)|\bar v|_{H_0^1}\right) \\ &= 0\left(\left(\tfrac{1}{\lambda^{1/2}} + \varepsilon\lambda^{1/2}\right)|\bar v|_{H_0^1}\right) \end{aligned}$$

since

$$(32) \qquad \left|\frac{\partial^2 P\delta_i}{\partial(x_i)_j\partial(x_i)_\ell}\right|_{H_0^1} = 0(\lambda^2)$$

and

$$(33) \qquad \langle\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}, \bar v\rangle_{H_0^1} = 0(\frac{|\bar v|_{H_0^1}}{\lambda^{1/2}})$$

(32) follows from an explicit computation, whereas (33) is more subtle and is proved in the next section.

These estimates, together with Proposition 2 and 3, show that with $v = \bar v$, (E) is equivalent to a new system (E')

$$(E')\left\{\begin{array}{l} \beta_i = \widetilde V_{\alpha_i}(\varepsilon,\alpha,\lambda,x) \\ K_2\varepsilon - K_\varepsilon\left(\frac{H(x_i,x_i)}{\lambda_i} - \sum_{\ell\neq i}\frac{G(x_i,x_\ell)}{\lambda_i^{1/2}\lambda_\ell^{1/2}}\right) = \widetilde V_{\lambda_i}(\varepsilon,\alpha,\lambda,x) \\ \frac{1}{\lambda^{1/2}}\frac{\partial H}{\partial a_j}(x_i,x_i) - \sum_{\ell\neq i}\frac{1}{\lambda_j^{1/2}}\frac{\partial G}{\partial a_j}(x_i,x_\ell) = \widetilde V_{(x_i)_j}(\varepsilon,\alpha,\lambda,x) \end{array}\right.$$

$$1 \le i \le k \qquad 1 \le j \le 3$$

where $\widetilde V_{\alpha_i}, \widetilde V_{\lambda_i}, \widetilde V_{(x_i)_j}$ are smooth functions which satisfy

$$\left\{\begin{array}{ll} \widetilde V_{\alpha_i}(\varepsilon,\alpha,\lambda,x) & = 0(|\beta|^2 + \frac{1}{\lambda} + \varepsilon\ell n\lambda + \varepsilon^2) \\ \widetilde V_{\alpha_i}(\varepsilon,\alpha,\lambda,x) & = 0(\frac{\ell n\lambda}{\lambda^3} + \varepsilon^2\ell n\lambda + |\beta|(\varepsilon + \frac{1}{\lambda})) \\ \widetilde V_{(x_i)_j}(\varepsilon,\alpha,\lambda,x) & = 0(\frac{1}{\lambda} + \varepsilon^2\lambda + \frac{|\beta|}{\lambda^{1/2}}) \end{array}\right.$$

We are now able to prove Theorem 1 for $N = 3$.

3.2 Proof of the first part of Theorem 1

The argument is quite similar to what happens for $n \geq 4$, we repeat it for convenience of the reader. u_ε, satisfying the assumptions of Theorem 1, writes as (19), with $m^\varepsilon = (\alpha^\varepsilon, \lambda^\varepsilon, \mathbf{x}^\varepsilon, v^\varepsilon)$ a solution of (E). As v^ε goes to zero in $H_0^1(\Omega)$, it follows from Proposition 2 that $v^\varepsilon = \bar{v}^\varepsilon$ for ε small enough. According to the previous subsection, (E_{α_i}) implies that

$$\beta_i^\varepsilon = 0(\frac{1}{\lambda^\varepsilon} + \varepsilon \ell n \lambda^\varepsilon + \varepsilon^2) \tag{34}$$

with $\beta_i^\varepsilon = \bar{\alpha} - \alpha_i^\varepsilon$. From (E_{λ_i}) we deduce that

$$K_2 \varepsilon - K_3 \left(\frac{H(x_i^\varepsilon, x_i^\varepsilon)}{\lambda_i^\varepsilon} - \sum_{\ell \neq i} \frac{G(x_i^\varepsilon, x_j^\varepsilon)}{(\lambda_i^\varepsilon)^{1/2} (\lambda_\ell^\varepsilon)^{1/2}} \right) = 0 \left(\frac{1}{(\lambda^\varepsilon)^2} + \frac{\varepsilon \ell n \lambda^\varepsilon}{\lambda^\varepsilon} + \varepsilon^2 \ell n \lambda^\varepsilon \right) .$$

Setting

$$\frac{1}{\lambda_i} = \frac{K_2}{K_3} \Lambda_i^2 \varepsilon \qquad \Lambda_i > 0 \tag{35}$$

we notice that, since

$$\lambda_i^\varepsilon \to +\infty \quad \varepsilon \ell n \lambda_i^\varepsilon \to 0 \qquad \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} < c_0$$

we have

$$(\Lambda_i^\varepsilon)^2 \varepsilon \to 0 \qquad \varepsilon \ell n \Lambda_i^\varepsilon \to 0 \qquad \frac{\Lambda_i^\varepsilon}{\Lambda_j^\varepsilon} < c_0$$

and (E_{λ_i}) provides us with the equation

$$1 - H(x_i^\varepsilon, x_i^\varepsilon)(\Lambda_i^\varepsilon)^2 + \sum_{\ell \neq i} G(x_i^\varepsilon, x_\ell^\varepsilon) \Lambda_i^\varepsilon \Lambda_\ell^\varepsilon = o(|\Lambda^\varepsilon|^2 + 1) . \tag{36}$$

Dividing each of these equations, $1 \leq i \leq k$, by Λ_i^ε respectively, we obtain

$$M(x^\varepsilon)^t \Lambda^\varepsilon + o(|\Lambda^\varepsilon|) = {}^t(\frac{1}{\Lambda^\varepsilon}) + o(\frac{1}{|\Lambda^\varepsilon|}) \tag{37}$$

with

$$\Lambda = (\Lambda_1, \cdots, \Lambda_k) \qquad \frac{1}{\Lambda} = (\frac{1}{\Lambda_1}, \cdots, \frac{1}{\Lambda_k}).$$

The scalar product of (37) with $r(x^\varepsilon)$, the unique unit vector with all its components strictly positive associated to the least, simple, eigenvalue of

$M(x^\varepsilon)$ (for a proof of the simplicity of ρ and that the components of an associated eigenvector have the same sign, see [3, Appendix A]), leads to

$$\rho(\mathbf{x}^\varepsilon)r(\mathbf{x}^\varepsilon).^t\Lambda^\varepsilon + o(|\Lambda^\varepsilon|) = r(\mathbf{x}^\varepsilon).^t(\frac{1}{\Lambda^\varepsilon}) + o(\frac{1}{|\Lambda^\varepsilon|}) \tag{38}$$

From $(E_{(x_i)_j})$ we deduce also

$$\frac{\partial H}{\partial a_j}(x_i^\varepsilon, x_i^\varepsilon)\Lambda_i^\varepsilon - \sum_{\ell \neq i} \frac{\partial G}{\partial a_j}(x_i^\varepsilon, x_\ell^\varepsilon)\Lambda_\ell^\varepsilon = 0\left(\varepsilon^{1/2}(|\Lambda^\varepsilon|^2 + \frac{1}{|\Lambda^\varepsilon|^2})\right) \tag{39}$$

Three cases may occur :

$$i)\Lambda^\varepsilon \to 0 \quad ii)\Lambda^\varepsilon \to \overline{\Lambda} \in (\mathbb{R}_+^*)^k \quad iii)\Lambda^\varepsilon \to +\infty (i.e.\ \Lambda_i^\varepsilon \to +\infty\ , \forall i)\ .$$

Actually, the first case is impossible, as (38) shows. Indeed, the left hand side would go to zero, since ρ is bounded from above on $\Omega_{d_0}^k$, and the right hand side would go to infinity. Considering the second case, we denote by $\bar{\mathbf{x}} \in \Omega_{d_0}^k$ the limit (ut to a subsequence) of $(\mathbf{x}^\varepsilon)$, and (38) yields

$$\rho(\bar{\mathbf{x}})r(\bar{\mathbf{x}})^t.\overline{\Lambda} = r(\bar{\mathbf{x}}).^t(\frac{1}{\overline{\Lambda}})$$

hence $\rho(\mathbf{x})$ is strictly positive. Moreover, the limit in (37) provides us with

$$M(\bar{\mathbf{x}})^t\overline{\Lambda} = \frac{1}{\overline{\Lambda}}\ .$$

This equality means that $\overline{\Lambda}$ is a critical point of $F_{\bar{\mathbf{x}}}$ in $(\mathbb{R}_+^*)^k$, i.e. $\overline{\Lambda} = \Lambda(\bar{\mathbf{x}})$ according to notations of Section 1, and the limit in (39) yields

$$\frac{\partial M}{\partial x_i}(\bar{\mathbf{x}})^t\Lambda(\bar{\mathbf{x}}) = 0$$

which implies, through (14)

$$\frac{\partial \widetilde{F}}{\partial x_i}(\bar{\mathbf{x}}) = 0\ .$$

Concerning the last case, it follows from (38) that $\rho(\mathbf{x}^\varepsilon)$ goes to zero, and then $\rho(\bar{\mathbf{x}}) = 0$. From (37) we have

$$M(\mathbf{x}^\varepsilon)^t\Lambda^\varepsilon = o(|\Lambda^\varepsilon|)$$

so that Λ^ε writes as

$$\Lambda^\varepsilon = \gamma^\varepsilon r(\mathbf{x}^\varepsilon) + \bar{r}^\varepsilon \tag{40}$$

with $\gamma^\varepsilon \to +\infty$, ${}^t\bar{r}^\varepsilon.r(\mathbf{x}^\varepsilon) = 0$, and $|\bar{r}^\varepsilon| = o(\gamma^\varepsilon)$. Indeed, since $\rho(\mathbf{x})$ is simple, the other eigenvalues are bounded from below far from zero as $\rho(\mathbf{x}^\varepsilon)$ goes to zero, and (40) follows from a decomposition of Λ^ε in a basis of eigenvectors for $M(x^\varepsilon)$. From (39) we know that

$$\frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon)^t\Lambda^\varepsilon = o(|\Lambda^\varepsilon|)$$

that is

$$\gamma^\varepsilon \frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon)^t r(\mathbf{x}^\varepsilon) + \frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon)^t\bar{r}^\varepsilon = o(\gamma^\varepsilon) \ .$$

The components of the matrix $\frac{\partial M}{\partial x_i}(\mathbf{x})$ are bounded on any set $\{\mathbf{x} \in \Omega^k_{d_0} / \rho \geq \rho_0\}$, for any $\rho_0 \in \mathbb{R}$. Therefore

$$\frac{\partial M}{\partial x_i}(\mathbf{x}^\varepsilon)^t r(\mathbf{x}^\varepsilon) = o(1) \ . \tag{41}$$

Derivating the equality $M(\mathbf{x}).{}^t r(\mathbf{x}) = \rho(\mathbf{x})^t r(\mathbf{x})$ with respect to x_i, we find

$$\frac{\partial M}{\partial x_i}(\mathbf{x})^t r(\mathbf{x}) + M(\mathbf{x})\frac{\partial^t r}{\partial x_i}(\mathbf{x}) = \frac{\partial \rho}{\partial x_i}(\mathbf{x})^t r(\mathbf{x}) + \rho(\mathbf{x})\frac{\partial^t r}{\partial x_i}(\mathbf{x}) \ .$$

Taking the scalar product with $r(\mathbf{x})$, this yields

$$r(\mathbf{x}).\frac{\partial M}{\partial x_i}(\mathbf{x})^t r(\mathbf{x}) = \frac{\partial \rho}{\partial x_i}(\mathbf{x})$$

and then (41) implies that

$$\frac{\partial \rho}{\partial x_i}(\bar{\mathbf{x}}) = 0 \ .$$

The first part of Theorem 1 is proved. We note that 0 being assumed to be a regular value for ρ, case iii) cannot occur. Moreover, there exists $\rho_0 > 0$ such that $\rho(\bar{\mathbf{x}}) \geq \rho_0$. Otherwise there would exist a sequence $(\mathbf{x}^n)$ in $\Omega^k_{d_0}$ such that $\rho(\mathbf{x}^n) > 0, \rho(\mathbf{x}^n) \to 0$ and $\widetilde{F}'(\mathbf{x}^n) = 0$. Up to a subsequence, we may assume that $\mathbf{x}^n$ converges to some limit $\bar{\mathbf{x}}$ in $\overline{\Omega^k_{d_0}}$. It follows from the definition of $\Lambda(\mathbf{x}^n)$ that

$$M(\mathbf{x}^n)^t\Lambda(\mathbf{x}^n) = {}^t(\frac{1}{\Lambda(\mathbf{x}^n)})$$

The scalar product with $r(\mathbf{x}^n)$ gives us the equality

$$\rho(\mathbf{x}^n) r(\mathbf{x}^n).{}^t\Lambda(\mathbf{x}^n) = r(\mathbf{x}^n).{}^t(\frac{1}{\Lambda}(\mathbf{x}^n))$$

Then $\rho(\mathbf{x}^n) \to 0$ implies that $\Lambda(\mathbf{x}^n) \to +\infty$, and the previous argument may be repeated, which shows that

$$\rho(\bar{\mathbf{x}}) = 0 \quad \rho'(\bar{\mathbf{x}}) = 0$$

a contradiction ; hence the announced result. The expansion of $J_\varepsilon(u^\varepsilon)$ given in Theorem 1 follows from direct computations, using (21) to estimate the v-part.

3.3 The converse part of Theorem 1.

Let $\bar{\mathbf{x}} \in \Omega^k$ such that $\rho(\bar{\mathbf{x}}) > 0$ and $\bar{\mathbf{x}}$ is a nondegenerate critical point of $\widetilde{F}$. We perform the changes of variables

$$\frac{1}{\lambda_i} = \frac{K_2}{K_3}(\Lambda_i(\bar{\mathbf{x}}) + \zeta_i)^2 \varepsilon \quad x_i = \bar{x}_i + \xi_i$$

$\zeta_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}^3$ being assumed to be small. With $v = \bar{v}(\varepsilon, \alpha, \lambda, \mathbf{x})$, the previous estimates show that solving (E) is equivalent to finding $(\beta, \zeta, \xi) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ such that

$$\beta_i = V_{\beta_i}(\varepsilon, \beta, \zeta, \xi) \tag{42}$$

$$\begin{aligned}(2H(\bar{x}_i, \bar{x}_i) - \textstyle\sum_{j\neq i} G(\bar{x}_i, \bar{x}_j)\overline{\Lambda}_j(\bar{\mathbf{x}}))\zeta_i - \textstyle\sum_{j\neq i} G(\bar{x}_i, \bar{x}_j)\overline{\Lambda}_i(\bar{\mathbf{x}})\zeta_j \\ + \left(2\overline{\Lambda}_i^2(\bar{\mathbf{x}})\nabla_a H(\bar{x}_i, \bar{x}_i) - \textstyle\sum_{j\neq i} \overline{\Lambda}_i(\bar{\mathbf{x}})\overline{\Lambda}_j(\bar{\mathbf{x}})\nabla_a G(\bar{x}_i, \bar{x}_j)\right).\xi_i \\ - \textstyle\sum_{j\neq i} \overline{\Lambda}_i(\bar{\mathbf{x}})\overline{\Lambda}_j(\bar{\mathbf{x}})\nabla_b G(\bar{x}_i, \bar{x}_j).\xi_j = V_{\zeta_i}(\varepsilon_1, \beta, \zeta, \xi)\end{aligned} \tag{43}$$

$$\begin{aligned}\nabla_a H(\bar{x}_i, \bar{x}_i)\zeta_i - \textstyle\sum_{j\neq i} \nabla_a G(\bar{x}_i, \bar{x}_j)\zeta_j + \Big(\overline{\Lambda}_i(\bar{\mathbf{x}})\nabla^2_{a^2} H(\bar{x}_i, \bar{x}_i) \\ + \overline{\Lambda}_i(\bar{\mathbf{x}})\nabla^2_{ab} H(\bar{x}_i, \bar{x}_i) - \textstyle\sum_{j\neq i} \overline{\Lambda}_j(\bar{\mathbf{x}})\nabla^2_{a^2} G(\bar{x}_i, \bar{x}_j)\Big).\xi_i \\ - \textstyle\sum_{j\neq i} \overline{\Lambda}_j(\bar{\mathbf{x}})\nabla^2_{ab} G(\bar{x}_i, \bar{x}_j).\xi_j = V_{\xi_i}(\varepsilon, \beta, \zeta, \xi)\end{aligned} \tag{44}$$

with $V_{\beta_i}, V_{\zeta_i}, V_{\xi_i}$ smooth functions which satisfy

$$\left\{\begin{array}{ll} V_{\beta_i} & = 0(\varepsilon|\ell n\varepsilon| + |\beta|^2) \\ V_{\zeta_i} & = 0(\varepsilon|\ell n\varepsilon| + |\beta| + |\zeta|^2 + |\xi|^2) \\ V_{\xi_i} & = 0(\varepsilon^{1/2} + |\beta| + |\xi|^2 + |\xi|^2) \end{array}\right. \tag{45}$$

Equations (42)(43)(44) may be written as

$$\left\{\begin{array}{ll} \beta & = V(\varepsilon,\beta,\zeta,\xi) \\ L(\zeta,\xi) & = W(\varepsilon,\beta,\zeta,\xi) \end{array}\right. \tag{46}$$

L being a fixed linear operator in $\mathbb{R}^k \times (\mathbb{R}^3)^k$ and V, W smooth functions which satisfy

$$\left\{\begin{array}{ll} V(\varepsilon,\beta,\zeta,\xi) & = 0(\varepsilon|\ell n\varepsilon| + |\beta|^2) \\ W(\varepsilon,\beta,\zeta,\xi) & = 0(\varepsilon^{1/2} + |\beta| + |\zeta|^2 + |\xi|^2) \; . \end{array}\right. \tag{47}$$

Moreover, the determinant of L is equal, up to a strictly positive number, to the determinant of $\widetilde{F}''(\bar{x})$, as an easy computation shows. $\bar{\mathbf{x}}$ being assumed to be a nondegenerate critical point of $\widetilde{F}$, L is invertible, and Brouwer's fixed point theorem ensures, provided that ε is small enough, the existence of a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ to (47), such that

$$|\beta^\varepsilon| = 0(\varepsilon|\ell n\varepsilon|) \quad |\zeta^\varepsilon| = 0(\varepsilon^{1/2}) \quad |\xi^\varepsilon| = 0(\varepsilon^{1/2}) \; .$$

By construction, $u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P\delta_{\lambda_i^\varepsilon, x_i^\varepsilon} + \bar{v}(\varepsilon, \alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon)$ with

$$\alpha_i^\varepsilon = \bar{\alpha} - \beta_i^\varepsilon \qquad \frac{1}{\lambda_i^\varepsilon} = \frac{K_2}{K_3}(\Lambda_i(\bar{\mathbf{x}}) + \zeta_i^\varepsilon)^2\varepsilon \qquad x_i^\varepsilon = \bar{x}_i + \xi_i^\varepsilon$$

is a critical point of J_ε, whence

$$-\Delta u_\varepsilon = |u_\varepsilon|^{4-\varepsilon} u_\varepsilon \text{ in } \Omega \; .$$

Multiplying this equation by $u_\varepsilon^- = \max(0, -u_\varepsilon)$ and integrating on Ω, one obtains

$$\int_\Omega |\nabla u_\varepsilon^-|^2 = \int_\Omega (u_\varepsilon^-)^{6-\varepsilon} \; . \tag{48}$$

From Sobolev embedding theorem we derive that

$$\left(\int_\Omega (u_\varepsilon^-)^{6-\varepsilon}\right)^{\frac{2}{6-\varepsilon}} \le C \int_\Omega |\nabla u_\varepsilon^-|^2 \; . \tag{49}$$

(48)(49) show that either $u_\varepsilon^- \equiv 0$, or $|u_\varepsilon^-|_{6-\varepsilon} \geq C^{-\frac{4-\varepsilon}{6-\varepsilon}} \geq C' > 0$. We know that $|u_\varepsilon^-|_{6-\varepsilon} \leq |\bar{v}^\varepsilon|_{6-\varepsilon}$ and $\bar{v}^\varepsilon$ goes to zero in $H_0^1(\Omega)$. Therefore, $u_\varepsilon^- \equiv 0$ for ε small enough, and u_ε satisfies

$$-\Delta u_\varepsilon = u_\varepsilon^{5-\varepsilon}, u_\varepsilon \geq 0 \text{ in } \Omega \ , \ \ u_\varepsilon = 0 \text{ on } \partial\Omega \ , u_\varepsilon \not\equiv 0 \ .$$

The strong maximum principle implies that $u_\varepsilon > 0$ in Ω, hence the desired result.

4 Proof of proposition 3

We concentrate our attention on formula (29), that is the derivative of the K_ε with respect to $(x_i)_j$. Formulas (27) and (28), concerning the derivatives of K_ε with respect to α_i and λ_i, may be obtained in the same way, with easier computations which moreover do not differ with the case $N \geq 4$. The method that we developp will also allow us to prove estimate (33).

In view of the definition of K_ε, we have

$$\begin{aligned}\frac{1}{\alpha_i}\frac{\partial K_\varepsilon}{\partial(x_i)_j}(\alpha,\lambda,\mathbf{x},\bar{v}) &= \int_\Omega \nabla(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell).\nabla\frac{\partial P\delta_i}{\partial(x_i)_j} \\ &\quad - \int_\Omega |\sum_{\ell=1}^k \alpha_\ell P\delta_\ell + \bar{v}|^{4-\varepsilon}(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell + \bar{v})\frac{\partial P\delta_i}{\partial(x_i)_j} \ .\end{aligned}$$

This may be written as

(50)
$$\begin{aligned}&\tfrac{1}{\alpha_i}\tfrac{\partial K_\varepsilon}{\partial(x_i)_j}(\alpha,\lambda,x,\bar{v}) = \textstyle\int_\Omega \nabla(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell).\nabla\frac{\partial P\delta_i}{\partial(x_i)_j} - \int_\Omega(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{5-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j} \\ &-(5-\varepsilon)\textstyle\int_\Omega(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{4-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j}\bar{v} - \frac{(-5-\varepsilon)(4-\varepsilon)}{2}\int_\Omega(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{3-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j}\bar{v}^2 \\ &+0\left(\textstyle\int_\Omega(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^2|\frac{\partial P\delta_i}{\partial(x_i)_j}||\bar{v}|^3 + \int_\Omega|\frac{\partial P\delta_i}{\partial(x_i)_j}||\bar{v}|^{5-\varepsilon}\right) .\end{aligned}$$

The terms where $\bar{v}$ does not occur may be computed explicitly, using (3)(4)(17), and one finds - see the integral estimates in [1][17] :

(51)
$$\begin{aligned}&\textstyle\int_\Omega \nabla(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell).\nabla\frac{\partial P\delta_i}{\partial(x_i)_j} - \int_\Omega(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{5-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j} \\ &= K_4\left(\tfrac{1}{\lambda_i}\tfrac{\partial H}{\partial a_j}(x_i,x_i) - \textstyle\sum_{\ell\neq i}\frac{1}{\lambda_i^{1/2}\lambda_\ell^{1/2}}\frac{\partial G}{\partial a_j}(x_i,x_\ell)\right) + 0(\tfrac{1}{\lambda^2} + \tfrac{\varepsilon \ell n\lambda}{\lambda} + \tfrac{|\beta|}{\lambda}) \ .\end{aligned}$$

It only remains to control the integrals involving $\bar{v}$. In particular, it is of crucial importance to obtain estimates of lower order than $1/\lambda$ as λ goes to infinity, in order to keep the previous term as a dominating term. The two last integrals in (50) are easy to treat. Namely, using Hölder's inequality and Sobolev embedding theorem, we have

$$\int_\Omega \left|\frac{\partial P\delta_i}{\partial(x_i)_j}\right| |\bar{v}|^{5-\varepsilon} = 0\left[\left(\int_\Omega |\frac{\partial P\delta_i}{\partial(x_i)_j}|^6\right)^{1/6} |\bar{v}|^{5-\varepsilon}_{H_0^1}\right] .$$

We note that, according to (3)

$$\frac{\partial \delta_{\lambda,x}}{\partial x_j}(y) = \frac{\lambda^{5/2}(y-x)_j}{(1+\lambda^2|x-y|^2)^{3/2}} \quad y \in \mathbb{R}^3 \tag{52}$$

and, according to (4)(15)(17) and the maximum principle

$$\frac{\partial \varphi_{\lambda,x}}{\partial x_j}(y) = 0(\frac{1}{\lambda^{1/2}}) \text{ uniformly in } \Omega \tag{53}$$

x being assumed to remain far from the boundary of Ω $\quad (d(x, \partial\Omega) > d_0)$. As a consequence, we have

$$\frac{\partial P\delta_i}{\partial x_i} = 0(\lambda_i \delta_i) \text{ uniformly in } \Omega . \tag{54}$$

$\int_{\mathbb{R}^N} \delta^6_{\lambda,x}(y)dy$ being a constant, depending on N only, it follows that

$$\int_\Omega \left|\frac{\partial P\delta_i}{\partial(x_i)_j}\right| |\bar{v}|^{5-\varepsilon} = 0(\lambda|\bar{v}|^{5-\varepsilon}_{H_0^1}) . \tag{55}$$

In the same way, writing $(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^2|\frac{\partial P\delta_j}{\partial(x_i)_j}| = 0(\lambda_i \sum_{\ell=1}^k \delta_\ell^3)$, we get

$$\int_\Omega (\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^2 \left|\frac{\partial P\delta_i}{\partial(x_i)_j}\right| \bar{v}^3 = 0(\lambda|\bar{v}|^3_{H_0^1}) . \tag{56}$$

Let us now consider the linear and the quadratic terms in $\bar{v}$. Outside $\bigcup_{\ell=1}^k B_\ell$, with $B_\ell = B(x_\ell, d), P\delta_\ell = 0(\frac{1}{\lambda^{1/2}})$ and $\frac{\partial P\delta_i}{\partial(x_i)_j} = 0(\frac{1}{\lambda^{1/2}})$. Therefore

$$\int_{(\bigcup_{\ell=1}^k B_\ell)^c} (\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{4-\varepsilon} \frac{\partial P\delta_i}{\partial(x_i)_j} \bar{v} = 0(\frac{1}{\lambda^{5/2}}|\bar{v}|_{H_0^1})$$

$$\int_{(\bigcup\limits_{\ell=1}^{k} B_\ell)^c} (\sum_{\ell=1}^{k} \alpha_\ell P\delta_\ell)^{3-\varepsilon} \frac{\partial P\delta_i}{\partial (x_i)_j} \bar{v}^2 = 0(\frac{1}{\lambda^2}|\bar{v}|^2_{H_0^1})$$

using Hölder's inequality and Sobolev embedding theorem. On B_q , $q \neq i$, $\sum\limits_{\ell=1}^{k} \alpha_\ell P\delta_\ell = 0(\delta_q)$ and $\frac{\partial P\delta_i}{\partial (x_i)_j} = 0(\frac{1}{\lambda^{1/2}})$, so that

$$\int_{B_q} (\sum_{\ell=1}^{k} \alpha_\ell P\delta_\ell)^{4-\varepsilon} \frac{\partial P\delta_i}{\partial (x_i)_j} \bar{v} = 0\left(\frac{1}{\lambda^{1/2}}(\int_{B_q} \delta_q^{24/5})^{5/6}|\bar{v}|_{H_0^1}\right)$$

$$\int_{B_q} (\sum_{\ell=1}^{k} \alpha_\ell P\delta_\ell)^{3-\varepsilon} \frac{\partial P\delta_i}{\partial (x_i)_j} \bar{v}^2 = 0\left(\frac{1}{\lambda^{1/2}}(\int_{B_q} \delta_q^{9/2})^{2/3}|\bar{v}|^2_{H_0^1}\right)$$

still using Hölder's inequality and Sobolev embedding theorem. A simple computation leads to

$$(57) \qquad \left(\int_{B_q} \delta_q^{24/5}\right)^{5/6} = 0(\frac{1}{\lambda^{1/2}}) \qquad\qquad \left(\int_{B_q} \delta_q^{9/2}\right)^{2/3} = 0(\frac{1}{\lambda^{1/2}})$$

whence

$$(58) \qquad \begin{cases} \int_{B_i^c}(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{4-\varepsilon} \frac{\partial P\delta_i}{\partial (x_i)_j}\bar{v} & = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda}) \\ \int_{B_i^c}(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{3-\varepsilon} \frac{\partial P\delta_i}{\partial (x_i)_j}\bar{v} & = 0(\frac{|\bar{v}|^2_{H_0^1}}{\lambda}) \ . \end{cases}$$

We turn now to the last and most delicate part, the integrals on B_i. We first note that

$$(59) \qquad \begin{cases} |\int_{B_q}(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{4-\varepsilon} \frac{\partial \varphi_i}{\partial (x_i)_j}\bar{v}| & \leq \frac{C}{\lambda^{1/2}} \int_{B_q} \delta_i^4|\bar{v}| = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda}) \\ |\int_{B_q}(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{3-\varepsilon} \frac{\partial \varphi_i}{\partial (x_i)_j}\bar{v}^2| & \leq \frac{C}{\lambda^{1/2}} \int_{B_q} \delta_i^3|\bar{v}|^2 = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda}) \end{cases}$$

proceeding as previously. In order to estimate the terms involving $\frac{\partial \delta_i}{\partial (x_i)_j}$, we write on B_i

$$(60) \qquad \sum_{\ell=1}^{k} \alpha_\ell P\delta_\ell = \alpha_i\delta_i + (\sum_{\ell\neq i} \alpha_\ell P\delta_\ell - \alpha_i\varphi_i)(x_i) + 0(\frac{|x-x_i|}{\lambda^{1/2}})$$

and

(61)

$$\begin{aligned} (\sum_{\ell=1}^k \alpha_\ell P\delta_\ell)^{4-\varepsilon} & = \alpha_i^{4-\varepsilon}\delta_i^{4-\varepsilon} + (4-\varepsilon)\alpha_i^{3-\varepsilon}(\sum_{\ell\neq i} \alpha_\ell P\delta_\ell - \alpha_i\varphi_i)(x_i)\delta_i^{3-\varepsilon} \\ & +0\left(\frac{\delta_i^3|x-x_i|}{\lambda^{1/2}} + \frac{\delta_i^2}{\lambda}\right) . \end{aligned}$$

Concerning the last quantities in (61), we have

$$\frac{1}{\lambda}\int_{B_i}\delta_i^2\left|\frac{\partial\delta_i}{\partial x_i}\right||\bar{v}| \le C\int_{B_i}\delta_i^3|\bar{v}| = 0\left[\left(\int_{B_i}\delta_i^{18/5}\right)^{5/6}|\bar{v}|_{H_0^1}\right]$$

and

$$\frac{1}{\lambda^{1/2}}\int_{B_i}\delta_i^3|x-x_i|\left|\frac{\partial\delta_i}{\partial x_i}\right||\bar{v}| \le C\lambda^{1/2}\int_{B_i}\delta_i^4|x-x_i|\,|\bar{v}|$$

$$= 0\left[\lambda^{1/2}\left(\int_{B_i}\delta_i^{24/5}|x-x_i|^{6/5}\right)^{5/6}|\bar{v}|_{H_0^1}\right].$$

Straightforward computation yield

$$(62)\qquad \left(\int_{B_i}\delta_i^{18/5}\right)^{5/6} = 0(\frac{1}{\lambda}) \qquad \left(\int_{B_i}\delta_i^{24/5}|x-x_i|^{6/5}\right)^{5/6} = 0(\frac{1}{\lambda^{3/2}})$$

so that the contribution of the last term in (61), multiplied by $\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v}$, to the integral on B_i, is dominated by $\frac{|\bar{v}|_{H_0^1}}{\lambda}$. Let us now compute the contribution of the first term. We recall that

$$-\Delta P\delta_{\lambda,x} = 3\delta_{\lambda,x}^5 \text{ in } \mathbb{R}^3$$

so that

$$-\Delta\frac{\partial P\delta_{\lambda,x}}{\partial x_j} = 15\delta_{\lambda,x}^4\frac{\partial\delta_{\lambda,x}}{\partial x_j}.$$

As $\bar{v}\in E_{\lambda,\mathbf{x}}$, we know that

$$\int_\Omega\nabla\frac{\partial P\delta_i}{\partial(x_i)_j}.\nabla\bar{v} = -\int_\Omega\Delta\frac{\partial P\delta_i}{\partial(x_i)_j}.\bar{v} = 15\int_\Omega\delta_i^4\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v} = 0$$

and

$$\int_\Omega\delta_i^{4-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v} = \int_\Omega\delta_i^4(\delta_i^{-\varepsilon}-\frac{1}{\lambda_i^{\varepsilon/2}})\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v}.$$

Therefore

$$(63)\qquad \begin{aligned}&\textstyle\int_\Omega(\sum_{\ell=1}^k\alpha_\ell P\delta_\ell)^{4-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j}\bar{v} = \alpha_i^{4-\varepsilon}\int_{B_i}\delta_i^4(\delta_i^{-\varepsilon}-\frac{1}{\lambda_i^{\varepsilon/2}})\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v}\\ &\textstyle+(4-\varepsilon)\alpha_i^{3-\varepsilon}(\sum_{\ell\ne i}\alpha_\ell P\delta_\ell-\alpha_i\varphi_i)(x_i)\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v}+0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda}\right)\end{aligned}$$

Concerning the quadratic term in $\bar{v}$, we write on B_i

$$\left(\sum_{\ell=1}^k\alpha_\ell P\delta_\ell\right)^{3-\varepsilon} = \alpha_i^{3-\varepsilon}\delta_i^{3-\varepsilon}+0(\frac{\delta_i^2}{\lambda^{1/2}}).$$

As

$$\frac{1}{\lambda^{1/2}}\int_{B_i}\delta_i^2|\frac{\partial\varphi_i}{\partial x_i}|\bar v^2 \le C\frac{1}{\lambda}(\int_{B_i}\delta_i^3)^{2/3}|\bar v|^2_{H_0^1} = 0\left(\frac{(\ell n\lambda)^{2/3}}{\lambda^2}|\bar v|^2_{H_0^1}\right)$$

$$\frac{1}{\lambda^{1/2}}\int_{B_i}\delta_i^2|\frac{\partial\varphi_i}{\partial x_i}|\bar v^2 \le C\frac{1}{\lambda}(\int_{B_i}\delta_i^{9/2})^{2/3}|\bar v|^2_{H_0^1} = 0\left(|\bar v|^2_{H_0^1}\right)$$

and

$$\int_{B_i}\delta_i^{3-\varepsilon}|\frac{\partial\varphi}{\partial x_i}|\bar v^2 \le \frac{C}{\lambda^{1/2}}(\int_{B_i}\delta_i^{9/2})^{2/3}|\bar v|^2_{H_0^1} = 0(\frac{|\bar v|^2_{H_0^1}}{\lambda})$$

we have

$$\int_\Omega\left(\sum_{\ell=1}^k\alpha_\ell P\delta_\ell\right)^{3-\varepsilon}\frac{\partial P\delta_i}{\partial(x_i)_j}\bar v^2 = \alpha_i^{3-\varepsilon}\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar v^2 + 0(|\bar v|^2_{H_0^1}) \tag{64}$$

Using, as previously, Hölder's inequality and Sobolev embedding theorem, and the fact that $\frac{\partial\delta_i}{\partial(x_i)_j} = 0(\lambda\delta_i)$, $\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} = 0(\varepsilon\ell n(1+\lambda_i^2|x-x_i|^2))$, we find, in view of (63) (64)

$$\left|\int_{B_i}\delta_i^4\left(\delta_i^{-\varepsilon}-\frac{1}{\lambda_i^{\varepsilon/2}}\right)\frac{\partial\delta_i}{\partial(x_i)_j}\bar v\right| \le C\varepsilon\lambda_i\left(\int_{B_i}\delta_i^6(\ell n(1+\lambda_i^2|x-x_i|^2))^{6/5}\right)^{5/6}|\bar v|_{H_0^1}$$

$$= 0(\varepsilon\lambda|\bar v|_{H_0^1})$$

$$\left|\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar v\right| \le C\lambda_i(\int_{B_i}\delta_i^{24/5})^{5/6}|\bar v|_{H_0^1} = 0(\lambda^{1/2}|\bar v|_{H_0^1})$$

$$\left|\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar v^2\right| \le C\lambda_i(\int_{B_i}\delta_i^6)^{2/3}|\bar v|^2_{H_0^1} = 0(\lambda|\bar v|^2_{H_0^1})$$

In view of (21), we obtain for the $\bar v$-part in the expansion of $\frac{\partial K_\varepsilon}{\partial(x_i)_j}$ a quantity dominated by $\frac{1}{\lambda}+\lambda\varepsilon^2$. This is sufficient to get (30), and therefore estimate (22) in Proposition 2. However, we need sharper estimates in (63) and (64) to prove (29).

For these purposes, we remark that δ_i is even and $\frac{\partial\delta_i}{\partial(x_i)_j}$ is odd with respect to the variable $(x-x_i)_j$. Splitting $\bar v$ in an even and an odd part with respect to this variable in a neighbourhood of x_i, we are able to obtain a better estimate than (21) concerning the odd part. As the even part has no contribution to the integrals, because of the oddness of $\frac{\partial\delta_i}{\partial(x_i)_j}$, the method will provide us with a suitable control of the quantities involving $\bar v$ and $\bar v^2$.

Let us make this precise. Firstly, we set

$$\bar{v} = \sum_{i=1}^{k} v_i + w \tag{65}$$

with v_i the projection of $\bar{v}$ onto $H_0^1(B_i)$, that is

$$\Delta v_i = \Delta \bar{v} \text{ in } B_i \ ; \ v_i = 0 \text{ on } \partial B_i \tag{66}$$

v_i being continued by 0 in $\Omega \setminus B_i$. Note that $w \in H_0^1(\Omega)$ is harmonic in B_i, and is orthogonal to v_i, that is

$$\Delta w = 0 \text{ in } B_i \ ; \ \int_\Omega \nabla w . \nabla v_i = 0 \qquad \forall i, 1 \le i \le k \ . \tag{67}$$

As a consequence, we have

$$\int_\Omega |\nabla \bar{v}|^2 = \sum_{i=1}^{k} \int_\Omega |\nabla v_i|^2 + \int_\Omega |\nabla w|^2 \ . \tag{68}$$

We split v_i in an even part v^e and an odd part v^o with respect to $(x - x_i)_j$. On B_i, $\bar{v} = v^e + v^o + w$, whence, in view of (63) and (64)

$$\int_{B_i} \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i}^{\varepsilon/2}) \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} = \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} (v^o + w)$$

$$= \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} \right) \frac{\partial \delta_i}{\partial (x_i)_j} w + 0(\varepsilon \lambda |v^o|_{H_0^1})$$

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v} = \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (v^o + w) = \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} w + 0(\lambda^{1/2} |v^o|_{H_0^1})$$

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} \bar{v}^2 = \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2 v^e v^o + 2(\bar{v} - w) w + w^2)$$

$$= \int_{B_i} \delta_i^{3-\varepsilon} \frac{\partial \delta_i}{\partial (x_i)_j} (2\bar{v} - w) w + 0(\lambda |\bar{v}|_{H_0^1} |v^o|_{H_0^1})$$

We claim that

$$
(69) \qquad |v^o|_{H_0^1} = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}\right) .
$$

estimate which will be sufficient to conclude. Concerning the integrals where w occurs, we use the fact that w is harmonic in B_i. Let ψ the solution of

$$
\Delta\psi = \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}\right) \frac{\partial \delta_i}{\partial (x_i)_j} \text{ in } B_i \ , \ \psi = 0 \text{ on } \partial B_i \ .
$$

Thus we have

$$
(70) \qquad \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}\right) \frac{\partial \delta_i}{\partial (x_i)_j} w = \int_{B_i} \Delta\psi . w = \int_{\partial B_i} \frac{\partial \psi}{\partial n} w \ .
$$

Let G_i the Green's function for the Laplacian on B_i, that is

$$
G_i(x,y) = \frac{1}{|x-y|} - \frac{d}{|x||y - \frac{d^2 x}{|x|^2}|} \quad (x,y) \in B_i^2 \ .
$$

ψ is given by

$$
\psi(y) = \int_{B_i} G_i(x,y) \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}) \frac{\partial \delta_i}{\partial (x_i)_j} dx \qquad y \in B_i
$$

and its normal derivative by

$$
\frac{\partial \psi}{\partial n}(y) = \int_{B_i} \frac{\partial G_i}{\partial n_y}(x,y) \left(\delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}) \frac{\partial \delta_i}{\partial (x_i)_j} \right)(x) dx \qquad y \in \partial B_i
$$

with

$$
\frac{\partial G_i}{\partial n_y}(x,y) = \frac{x.y - d^2}{d|x-y|^3} + d^2 \frac{|x|^2 - x.y}{|x|^3 |y - \frac{d^2 x}{|x|^2}|} = 0(\frac{1}{|x-y|^2}) \ .
$$

For $x \in B_i \setminus B(y, \frac{d}{2})$

$$
\frac{\partial G_i}{\partial n_y}(x,y) = 0(1) \qquad \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}) \frac{\partial \delta_i}{\partial (x_i)_j}(x) = 0(\varepsilon \lambda \delta_i^5 \ell n (1 + \lambda_i^2 |x - x_i|^2))
$$

and for $x \in B_i \cap B(y, \frac{d}{2})$

$$
\frac{\partial G_i}{\partial n_y}(x,y) = 0\left(\frac{1}{|x-y|^2}\right) \qquad \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}}) \frac{\partial \delta_j}{\partial (x_i)_j}(x) = 0(\frac{\varepsilon \ell n \lambda}{\lambda^{3/2}}) \ .
$$

Therefore

$$\frac{\partial\psi}{\partial n}(y) = 0\left(\varepsilon\lambda\int_{B_i}\delta_i^5\ell n(1+\lambda_i^2|x-x_i|^2)dx + \frac{\varepsilon\ell n\lambda}{\lambda^{5/2}}\int_{|x|\leq\frac{d}{2}}\frac{dx}{|x|^2}\right)$$
$$= 0(\varepsilon\lambda^{1/2})$$

and

$$\int_{\partial B_i}\frac{\partial\psi}{\partial n}w = 0(\varepsilon\lambda^{1/2}\int_{\partial B_i}|w|)$$

so that, from (70) we deduce

$$\int_{B_i}\delta_i^4\left(\delta_i^{-\varepsilon}-\frac{1}{\lambda_i^{\varepsilon/2}}\right)\frac{\partial\delta_i}{\partial(x_i)_j}w = 0(\varepsilon\lambda^{1/2}|\bar{v}|_{H_0^1})\ . \tag{71}$$

Concerning the second integral in which w occurs, we set ψ' as

$$\Delta\psi' = \delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\ \text{in } B_i\ ;\psi' = 0 \text{ on } \partial B_i$$

and we may write

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}w = \int_{\partial B_i}\frac{\partial\psi'}{\partial n}w\ .$$

Proceeding as previousy, we have, for $y\in\partial B_i$

$$\frac{\partial\psi'}{\partial n}(y) = 0\left(\lambda\int_{B_i}\delta_i^4+\frac{1}{\lambda^2}\int_{|x|\leq\frac{d}{2}}\frac{dx}{|x|^2}\right)$$
$$= 0(1)$$

and

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}w = 0(|\bar{v}|_{H_0^1})\ . \tag{72}$$

Lastly, we define ψ'' as

$$\Delta\psi'' = \delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}(2\bar{v}-w)\ \text{in } B_i\ ;\psi'' = 0 \text{ on } \partial B_i$$

and we write

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}(2\bar{v}-w)w = \int_{\partial B_i}\frac{\partial\psi''}{\partial n}w\ .$$

Note that, as $2\bar{v} - w \in W^{1,2}(B_i), \psi'' \in W^{3,2}(B_i)$, and $\nabla\psi''_{|\partial B_i} \in L^\infty(\partial B_i)$. Actually, the integral definition of ψ'' yields

$$\begin{aligned}
\frac{\partial\psi''}{\partial n}(y) &= 0\left(\lambda\int_{B_i}\delta_i^4|2\bar{v}-w| + \frac{1}{\lambda^2}\int_{B(y,\frac{d}{2})}|(2\bar{v}-w)(x)|\frac{dx}{|x-y|^2}\right)\\
&= 0\left(\lambda|\bar{v}|_{H_0^1}(\int_{B_i}\delta_i^{24/5})^{5/6} + \frac{1}{\lambda^2}|\bar{v}|_{H_0^1}\right)\\
&= 0(\lambda^{1/2}|\bar{v}|_{H_0^1})
\end{aligned}$$

uniformly for $y \in \partial B_i$. Therefore

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}(2\bar{v}-w)w = 0(\lambda^{1/2}|\bar{v}|^2_{H_0^1}) \ . \tag{73}$$

(71) (72) (73) and claim (69), together with (21), show that

$$\int_{B_i}\delta_i^4(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}})\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v} = 0(\frac{\varepsilon}{\lambda^{1/2}} + \lambda^{1/2}\varepsilon^2)$$

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v} = 0(\frac{1}{\lambda} + \varepsilon)$$

$$\int_{B_i}\delta_i^{3-\varepsilon}\frac{\partial\delta_i}{\partial(x_i)_j}\bar{v}^2 = 0(\frac{1}{\lambda^{3/2}} + \lambda^{1/2}\varepsilon) \ .$$

Then, from (50) (51) (55) (56) (58) (63) (64) and (21) we deduce (29), that is the desired estimate of $\frac{\partial K_\varepsilon}{\partial(x_i)_j}$.

We are also able to derive (33) from the above estimates. Namely, we have

$$\begin{aligned}
\langle\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}, \bar{v}\rangle_{H_0^1} &= \int_\Omega \nabla\frac{\partial^2 P\delta_i}{\partial\lambda_i\partial(x_i)_j}.\nabla\bar{v}\\
&= \int_\Omega(60\delta_i^3\frac{\partial\delta_i}{\partial\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_j} + 15\delta_i^4\frac{\partial^2\delta_i}{\partial\lambda_i\partial(x_i)_j})(v^e + v^o + w) \ .
\end{aligned}$$

The contribution of the subdomain $\Omega\backslash B_i$ to the integral is very small. Indeed, outside B_i

$$\delta_i^3\frac{\partial\delta_i}{\partial\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_j} = 0(\frac{1}{\lambda^{7/2}}) \qquad \delta_i^4\frac{\partial^2\partial_i}{\partial\lambda_i\partial(x_i)_j} = 0(\frac{1}{\lambda^{7/2}}) \ .$$

On B_i, the contribution of v^e is zero because of evenness with respect to the first variable and, as $\frac{\partial \delta_i}{\partial \lambda_i} = 0(\frac{\delta_i}{\lambda_i})$, $\frac{\partial \delta_i}{\partial (x_i)_j} = 0(\lambda_i \delta_i)$, $\frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} = 0(\delta_i)$

$$\left| \int_{B_i} \left(60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_i^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \right) v^o \right| \leq C|v^o|_{H_0^1} = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}})$$

according to (69). Lastly

$$\int_{B_i} \left(60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_i^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \right) w = \int_{B_i} \Delta \psi'''.w = \int_{\partial B_i} \frac{\partial \psi'''}{\partial n} w$$

with

$$\Delta \psi''' = 60\delta_i^3 \frac{\partial \delta_i}{\partial \lambda_i} \frac{\partial \delta_i}{\partial (x_i)_j} + 15\delta_1^4 \frac{\partial^2 \delta_i}{\partial \lambda_i \partial (x_i)_j} \text{ in } B_i \text{ ; } \psi''' = 0 \text{ on } \partial B_i \text{ .}$$

For $y \in \partial B_i$, $\frac{\partial \psi'''}{\partial n}$ satisfies

$$|\frac{\partial \psi'''}{\partial n}(y)| \leq C \int_{B_i} \left| \frac{\partial G}{\partial n_y}(x,y) \right| \delta_i^5(x) dx = 0(\frac{1}{\lambda^{1/2}}) \text{ .}$$

As a consequence

$$\int_{\partial B_i} \frac{\partial \psi'''}{\partial n} w = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}})$$

and (33) follows.

Proof of Claim (69)

The estimate of $|v^o|_{H_0^1}$ will make the proof of Proposition 3 complete. For sake of simplicity, we may assume that $i = j = 1$ and, up to a translation, that $x_1 = 0$. We write

$$v^o = \tilde{v}^o + aP\delta_1 + b\frac{\partial P\delta_1}{\partial \lambda_1} + \sum_{\ell=1}^{3} c_\ell \frac{\partial P\delta_1}{\partial (x_1)_\ell} \tag{74}$$

with

$$\langle P\delta_1, \tilde{v}^o \rangle_{H_0^1} = \langle \frac{\partial P\delta_1}{\partial \lambda_1}, \tilde{v}^o \rangle_{H_0^1} = \langle \frac{\partial P\delta_1}{\partial (x_i)_\ell}, \tilde{v}^o \rangle_{H_0^1} = 0 \quad 1 \leq \ell \leq 3 \text{ .}$$

Taking the scalar product in $H_0^1(\Omega)$ of (74) with $P\delta_1$, $\frac{\partial P\delta_1}{\partial \lambda_1}$, $\frac{\partial P\delta_1}{\partial (x_i)_\ell}$, $1 \leq \ell \leq 3$, provides us with an invertible linear system in a, b, c_ℓ, whose coefficients are given by (25) (26). On the left hand side, we find

$$\int_{B_1} \nabla P\delta_1 \,.\, \nabla v^o = 0 \tag{75}$$

since

$$\int_{B_1} \nabla\delta_1 \,.\, \nabla v^o = 0$$

because of evenness of δ_1 and oddness of v^o with respect to the first variable, and

$$\int_{B_1} \nabla\varphi_1 . \nabla v^o = 0$$

because of harmonicity of φ_1 and nullity of v^o on ∂B_1. In the same way

$$\int_{B_1} \nabla \frac{\partial P\delta_1}{\partial \lambda_1} . \nabla v^o = \int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_\ell} . \nabla v^o = 0 \qquad \ell = 2, 3 \,. \tag{76}$$

Lastly, we have

$$\begin{aligned}\int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla v^o &= \int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla(\bar{v} - v^e - w) \\ &= - \int_{\Omega\backslash B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} \,. \nabla\bar{v} - \int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla w\end{aligned}$$

since $\bar{v} \in E_{\lambda,\mathbf{x}}$ and v^e is even with respect to the first variable, zero on ∂B_1, and $\frac{\partial \varphi}{\partial x_1}$ is harmonic in B_1. On one hand

$$\int_{\Omega\backslash B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla\bar{v} \leq \frac{C}{\lambda^{1/2}} \int_\Omega |\nabla\bar{v}| = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}) \,.$$

On the other hand, let $\psi^{(4)}$ be such that

$$\Delta\psi^{(4)} = \Delta \frac{\partial P\delta_1}{\partial (x_1)_1} = -15\delta_1^4 \frac{\partial \delta_i}{\partial (x_1)_1} \text{ in } B_1 \,; \psi^{(4)} = 0 \text{ on } \partial B_1 \,.$$

Writing

$$\psi^{(4)} = \frac{\partial P\delta_1}{\partial (x_1)_1} + \theta$$

we have

$$\begin{aligned}\int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla w &= \int_{B_1} \nabla(\psi^{(4)} - \theta) . \nabla w \\ &= - \int_{\partial B_1} \frac{\partial \theta}{\partial n} . w\end{aligned}$$

since w and θ are harmonic in B_1, and $\psi^{(4)}$ is zero on ∂B_1. Using, as previously, an integral representation for $\psi^{(4)}$, we obtain for $y \in \partial B_1$

$$\frac{\partial \psi^{(4)}}{\partial n}(y) = -15 \int_{B_1} \frac{\partial G_1}{\partial n_y}(x,y) \left(\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1} \right)(x)dx$$
$$= -15 \int_{B(0,\frac{d}{2})} \left(\frac{\partial G_1}{\partial n_y}(0,y) + 0(|x|) \right) (\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1})(x)dx + 0(\frac{1}{\lambda^{5/2}})$$

since $\nabla_x \frac{\partial G_1}{\partial n_y}(x,y)$ is bounded in $B(0,\frac{d}{2}) \times \partial B_1$, $\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1} = 0(\frac{1}{\lambda^{5/2}})$ in $B_1 \setminus B(0,\frac{d}{2})$ and $\frac{\partial G_1}{\partial n_y}(x,y) = 0(\frac{1}{|x-y|^2})$ in $B_1 \times \partial B_1$. $\delta_1^4 \frac{\partial \delta_1}{\partial (x_1)_1}$ being odd with respect to the first variable, and

$$\int_{B(0,\frac{d}{2})} |x| \delta_1^4 \left| \frac{\partial \delta_1}{\partial (x_1)_1} \right| dx = 0(\frac{1}{\lambda^{1/2}})$$

we find

$$\frac{\partial \psi^{(4)}}{\partial n}(y) = 0(\frac{1}{\lambda^{1/2}})$$

so that, noticing that $\frac{\partial}{\partial n}(\frac{\partial P\delta_1}{\partial (x_1)_1}) = 0(\frac{1}{\lambda^{1/2}})$ on ∂B_1

$$\frac{\partial \theta}{\partial n} = 0(\frac{1}{\lambda^{1/2}}) \text{ on } \partial B_1 .$$

It follows that

$$\int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla w = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}})$$

and

$$\int_{B_1} \nabla \frac{\partial P\delta_1}{\partial (x_1)_1} . \nabla v^o = 0(\frac{|\bar{v}|_{H_0^1}}{\lambda^{1/2}}) . \tag{77}$$

Inverting the linear system involving a, b, c_ℓ, whose coefficients are given by (25) (26) and whose left hand side is given by (75) (76) (77), the following estimates are obtained :

(78)
$$a = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{7/2}}\right) \quad b = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}}\right) \quad c_1 = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}}\right) \quad c_\ell = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{7/2}}\right) \ell = 2,3 .$$

Using again (25) (26), this implies through (74)

$$|v^o - \tilde{v}^o|_{H_0^1} = 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{3/2}}\right) \quad |v^o|^2_{H_0^1} = |\tilde{v}^o|^2_{H_0^1} + 0\left(\frac{|\bar{v}|^2_{H_0^1}}{\lambda^3}\right) . \tag{79}$$

We turn now to the last step, which consists in estimating $\tilde{v}^o$ in $H_0^1(\Omega)$. The scalar product of (E_v) with v^o yields the equality

(80)

$$\int_\Omega \nabla(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}).\nabla v^o - \int_\Omega |\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}|^{4-\varepsilon}(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v})v^o$$

$$= \int_\Omega \nabla \sum_{i=1}^k (A_i P\delta_i + B_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1}^3 C_{ij}\frac{\partial P\delta_i}{\partial (x_i)_j}).\nabla(aP\delta_1 + b\frac{\partial P\delta_1}{\partial \lambda_i} + \sum_{j=1}^3 c_j \frac{\partial P\delta_1}{\partial (x_1)_j})$$

whence

$$\int_\Omega \nabla(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}).\nabla v^o - \int_\Omega |\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}|^{4-\varepsilon}(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v})v^o = 0\left(|\bar{v}|_{H_0^1}\left(\frac{1}{\lambda^{5/2}} + \frac{\varepsilon^2}{\lambda^{1/2}}\right)\right) \tag{81}$$

using (22) (25) (26) and (78). Concerning the first integral, we know that

$$\int_\Omega \nabla(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}).\nabla v^o = -3\sum_{i=2}^k \alpha_i \int_{B_1} \delta_i^5 v^o + \int_{B_1} |\nabla v^o|^2$$

since $-\Delta P\delta_i = 3\delta_i^5$ in Ω, v^o is zero in $\Omega \setminus B_1$, $\bar{v} = v^e + v^o + w$ in B_1 with v^e even and v^o odd with respect to the first variable, and w harmonic in B_1. Therefore, as $\delta_i^5 = 0(\frac{1}{\lambda^{5/2}})$ in B_1 for $2 \le i \le k$, we find, taking account of (79) :

$$\int_\Omega \nabla\left(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}\right).\nabla v^o = \int_{B_1} |\nabla \tilde{v}^o|^2 + 0\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{5/2}}\right) \tag{82}$$

Let us consider the second integral, which may be restricted to B_1, since v^o is zero in $B_1 \setminus \Omega$. We expand

$$\int_{B_1} |\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}|^{4-\varepsilon}\left(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}\right) v^o$$

$$= \alpha_1^{5-\varepsilon}\int_{B_1} P\delta_1^{5-\varepsilon} v^o + (5-\varepsilon)\alpha_1^{4-\varepsilon}\int_{B_1} P\delta_1^{4-\varepsilon}\left(\sum_{i=2}^k \alpha_i P\delta_i + \bar{v}\right) v^o$$

$$+ 0\left[\int_{B_1}\left(\delta_1^3(\sum_{i=2}^k \delta_i^2 + |\bar{v}|^2) + \sum_{i=2}^k \delta_i^5 + |\bar{v}|^{5-\varepsilon}\right)|v^o|\right].$$

Estimating the last term is easy, namely

$$\int_{B_1}\left(\delta_1^3\left(\sum_{i=2}^k \delta_i^2+|\bar v|^2\right)+\sum_{i=2}^k \delta_i^5+|\bar v|^{5-\varepsilon}\right)|v^o|$$
$$=0\left[\frac{1}{\lambda}\left(\int_{B_1}\delta_1^{18/5}\right)^{5/6}|v^o|_{H_0^1}+|\bar v|^2_{H_0^1}|v^o|+\frac{1}{\lambda^{5/2}}|v^o|_{H_0^1}+|\bar v|^{5-\varepsilon}_{H_0^1}|v^o|_{H_0^1}\right]$$
$$=0\left[|v^o|_{H_0^1}\left(\frac{1}{\lambda^2}+|\bar v|^2_{H_0^1}\right)\right].$$

Concerning the remaining terms, we write

$$\int_{B_1}P\delta_1^{5-\varepsilon}v^o=\int_{B_1}\delta_1^{5-\varepsilon}v^o-(5-\varepsilon)\int_{B_1}\delta_1^{4-\varepsilon}\left(\varphi_1(0)+0(\frac{|x|}{\lambda^{1/2}})\right)v^o$$
$$+0\left(\frac{1}{\lambda}\int_{B_1}\delta_1^3|v^o|\right).$$

Using evenness of δ_1 and oddness of v^o with respect to the first variable, and noticing that

$$\left(\int_{B_1}\delta_1^{24/5}|x|^{6/5}\right)^{5/6}=0\left(\frac{1}{\lambda^{3/2}}\right)\qquad\left(\int_{B_1}\delta_i^{18/5}\right)^{5/6}=0(\frac{1}{\lambda})$$

we obtain

(83) $$\int_{B_1}P\delta_1^{5-\varepsilon}v^o=0\left(\frac{|v^o|_{H_0^1}}{\lambda^2}\right).$$

In the same way, we have

$$\int_{B_1}P\delta_1^{4-\varepsilon}\left(\sum_{i=2}^k\alpha_iP\delta_i\right)v^o$$
$$=\int_{B_1}\left(\delta_1^{4-\varepsilon}+0(\frac{\delta_1^3}{\lambda^{1/2}})\right)\left(\sum_{i=2}^k\alpha_iP\delta_i(x_1)+0(\frac{|x|}{\lambda^{1/2}})\right)v^o$$
$$=0\left(\frac{1}{\lambda^{1/2}}\int_{B_1}\delta_1^4|x||v^o|+\frac{1}{\lambda}\int_{B_1}\delta_1^3|v^o|\right)$$

whence

$$(84) \qquad \int_{B_1} P\delta_1^{4-\varepsilon}\left(\sum_{i=2}^{k}\alpha_i P\delta_i\right) v^o = 0\left(\frac{|v^o|_{H_0^1}}{\lambda^{3/2}}\right) .$$

The last term to consider writes as

$$(5-\varepsilon)\alpha_1^{4-\varepsilon}\int_{B_1} P\delta_1^{4-\varepsilon}\bar{v}v^o = (15+0(\varepsilon))\int_{B_1} P\delta_1^{4-\varepsilon}(v^e+v^o+w)v^o .$$

On one hand

$$\int_{B_1} P\delta_1^{4-\varepsilon}(v^e+v^o)v^o$$
$$= \int_{B_1}\left[\delta_1^{4-\varepsilon}-(4-\varepsilon)\delta_1^{3-\varepsilon}(\varphi_1(0)+0(\frac{|x|}{\lambda^{1/2}}))+0(\frac{\delta_i^2}{\lambda})\right](v^e+v^o)v^o$$
$$= \int_{B_1}\delta_1^{4-\varepsilon}(v^o)^2+0\Big(\frac{1}{\lambda^{1/2}}\int_{B_1}\delta_1^3(v^o)^2+\frac{1}{\lambda^{1/2}}\int_{B_1}\delta_1^3|x||v_1||v^o|+\frac{1}{\lambda}\int_{B_1}\delta_1^2|v_1||v^o|\Big)$$
$$= \int_{B_1}\delta_1^4(v^o)^2+0\Big(\varepsilon\ell n\lambda|v^o|^2_{H_0^1}+\frac{1}{\lambda^{1/2}}(\int_{B_1}\delta_1^{9/2})^{2/3}|v^o|^2_{H_0^1}$$
$$+\frac{1}{\lambda^{1/2}}(\int_{B_1}\delta_1^{9/2}|x|^{3/2})^{2/3}|\bar{v}|_{H_0^1}|v^o|_{H_0^1}+\frac{1}{\lambda}(\int_{B_1}\delta_1^3)^{2/3}|\bar{v}|_{H_0^1}|v^o|_{H_0^1}\Big)$$
$$= \int_{B_1}\delta_1^4(v^o)^2+o(|v^o|^2_{H_0^1})+0(\frac{(\ell n\lambda)^{2/3}}{\lambda^2}|\bar{v}|_{H_0^1}|v^o|_{H_0^1})$$

because of (57) and

$$\left(\int_{B_1}\delta_1^{9/2}|x|^{3/2}\right)^{2/3}=0\left(\frac{(\ell n\lambda)^{2/3}}{\lambda^2}\right) \quad \left(\int_{B_1}\delta_1^3\right)^{2/3}=0\left(\frac{(\ell n\lambda)^{2/3}}{\lambda}\right) .$$

On the other hand

$$\int_{B_1} P\delta_1^{4-\varepsilon}wv^o = \int_{B_1}\Delta\psi^{(5)}.w = \int_{\partial B_1}\frac{\partial\psi^{(5)}}{\partial n}w$$

with $\psi^{(5)}$ defined as

$$\Delta\psi^{(5)} = P\delta_1^{4-\varepsilon}v^o \text{ in } B_1\ ; \psi^{(5)}=0 \text{ on } \partial B_1 .$$

The normal derivative of $\psi^{(5)}$ at $y \in \partial B_1$ is given by

$$\begin{aligned}
\frac{\partial \psi^{(5)}}{\partial n}(y) &= \int_{B_1} \frac{\partial G_1}{\partial n_y}(x,y) P\delta_1^{4-\varepsilon}(x) v^o(x) dx \\
&= 0\left[\int_{\substack{B_1 \\ |x-y|\geq \frac{d}{2}}} \delta_1^4 |v^o| dx + \frac{1}{\lambda^2}\int_{\substack{B_1 \\ |x-y|\geq \frac{d}{2}}} |v^o| \frac{dx}{|x-y|^2}\right] \\
&= 0\left[\left(\int_{B_1} \delta_1^{24/5}\right)^{5/6} |v^o|_{H_0^1} + \frac{|v^o|_{H_0^1}}{\lambda^2}\right] \\
&= 0\left(\frac{|v^o|_{H_0^1}}{\lambda^{1/2}}\right)
\end{aligned}$$

because of (57). Consequently

$$\int_{B_1} P\delta_1^{4-\varepsilon} w v^o = 0\left(\frac{|v^o|_{H_0^1}}{\lambda^{1/2}} \int_{\partial B_1} |w|\right)$$

and so

$$\int_{B_1} P\delta_1^{4-\varepsilon} w v^o = 0\left(\frac{|\bar{v}|_{H_0^1} |v^o|_{H_0^1}}{\lambda^{1/2}}\right) .$$

This yields finally, taking account of (82)(83)(84)

$$\begin{aligned}
&\int_\Omega \nabla\Big(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}\Big).\nabla v^o - \int_\Omega \Big|\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}\Big|^{4-\varepsilon} \left(\sum_{i=1}^k \alpha_i P\delta_i + \bar{v}\right) v^o \\
&\quad = \int_{B_1} |\nabla v^o|^2 - 15\int_{B_1} \delta_1^4 (v^o)^2 + o\left(|v^o|^2_{H_0^1}\right) + 0\left(\frac{|\bar{v}|_{H_0^1}|v^o|_{H_0^1}}{\lambda^{1/2}}\right) \\
&\quad = \int_\Omega |\nabla \tilde{v}^o|^2 - 15\int_\Omega \delta_1^4 (\tilde{v}^o)^2 + o\left(|\tilde{v}^o|^2_{H_0^1}\right) + 0\left(\frac{|\bar{v}|_{H_0^1}|\tilde{v}^o|_{H_0^1}}{\lambda^{1/2}} + \frac{|\bar{v}|^2_{H_0^1}}{\lambda^2}\right)
\end{aligned}$$

because of (79). Comparing with (81) and (79), and the quadratic form

$$v \mapsto \int_\Omega |\nabla v|^2 - 15\int_\Omega \delta_1^4 v^2$$

being coercive on the subset $\left[\mathrm{Span}(P\delta_1, \frac{\partial P\delta_1}{\partial \lambda_1}, \frac{\partial P\delta_1}{\partial (x_1)_j}, 1 \leq j \leq 3)\right]^{\perp}_{H_0^1}$, (69) follows.

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